Pre-Algebra

Course No. 1200300

Bureau of Exceptional Education and Student Services
Florida Department of Education

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Curriculum Improvement Project
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LEON COUNTY SCHOOLS

Exceptional Student Education

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Unit 1: Number Sense, Concepts, and Operations

This unit emphasizes the effects of various operations on the real number system to solve problems.

Unit Focus

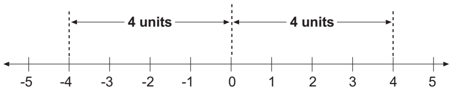
Number Sense, Concepts, and Operations

- Associate verbal names, written word names, and standard numerals with integers, rational numbers, irrational numbers, and real numbers. (MA.A.1.4.1)
- Understand the relative size of integers, rational numbers, and real numbers. (MA.A.1.4.2)
- Understand concrete and symbolic representations of real numbers in real-world situations. (MA.A.1.4.3)
- Understand that numbers can be represented in a variety of equivalent forms, including integers, fractions, decimals, percents, exponents, radicals, and absolute value. (MA.A.1.4.4)
- Understand and use the real number system. (MA.A.2.4.2)
- Understand and explain the effects of addition, subtraction, multiplication, and division on real numbers, including square roots, exponents, and appropriate inverse relationships. (MA.A.3.4.1)
- Select and justify alternative strategies, such as using properties of numbers, including inverse, identity, and associative, that allow operational shortcuts for computational procedures in real-world or mathematical problems. (MA.A.3.4.2)
- Add, subtract, multiply, and divide real numbers, including square roots and exponents, using appropriate methods of computing, such as mental mathematics, paper and pencil, and calculator. (MA.A.3.4.3)

- Use estimation strategies in complex situations to predict results and to check the reasonableness of results. (MA.A.4.4.1)

Vocabulary

Use the vocabulary words and definitions below as a reference for this unit.

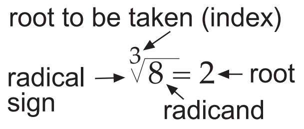
absolute value	a number's distance from zero (0) on the number line *Example*: The absolute value of both 4, written $\lvert 4 \rvert$, and negative 4, written $\lvert -4 \rvert$, equals 4. 
additive identity	the number zero (0), that is, adding 0 does not change a number's value *Example*: 5 + 0 = 5
additive inverses	a number and its opposite whose sum is zero (0); also called *opposites* *Example*: In the equation 3 + -3 = 0, 3 and -3 are additive inverses, or *opposites*, of each other.
area (*A*)	the inside region of a two-dimensional figure measured in square units *Example*: A rectangle with sides of four units by six units contains 24 square units or has an area of 24 square units.
associative property	the way in which three or more numbers are grouped for addition or multiplication does *not* change their sum or product *Example*: (5 + 6) + 9 = 5 + (6 + 9) *or* (2 x 3) x 8 = 2 x (3 x 8)
base (of an exponent)	the number that is used as a factor a given number of times *Example*: In 2^3, 2 is the base and 3 is the exponent.

commutative property	the order in which any two numbers are added or multiplied does *not* change their sum or product *Example*: 2 + 3 = 3 + 2 *or* 4 x 7 = 7 x 4
coordinate	the number paired with a point on the number line
cube (power)	the third power of a number *Example*: 4^3 = 4 x 4 x 4 = 64
decrease	to make less
difference	the result of a subtraction *Example*: In 16 – 9 = 7, 7 is the difference.
equation	a mathematical sentence that equates one expression to another expression *Example*: $2x = 10$
equivalent (forms of a number)	the same number expressed in different forms *Example*: $\frac{3}{4}$, 0.75, and 75%
estimation	the use of rounding and/or other strategies to determine a reasonably accurate approximation without calculating an exact answer
exponent (exponential form)	the number of times the base occurs as a factor *Example*: 2^3 is the exponential form of 2 x 2 x 2. The numeral two (2) is called the *base*, and the numeral three (3) is called the *exponent*.

expression	a collection of numbers, symbols, and/or operation signs that stands for a number *Example*: $4r^2$; $3x + 2y$; $\sqrt{25}$ Expressions do *not* contain equality (=) or inequality (<, >, ≤, ≥, or ≠) symbols.
factor	a number or expression that divides exactly another number *Example*: 1, 2, 4, 5, 10, and 20 are factors of 20.
fraction	any number representing some part of a whole; of the form $\frac{a}{b}$ *Example*: One-half written in fractional form is $\frac{1}{2}$.
graph of a number	the point on a number line paired with the number
increase	to make greater
inequality	a sentence that states one expression is greater than (>), greater than or equal to (≥), less than (<), less than or equal to (≤), or not equal to (≠) another expression *Example*: $a \neq 5$ or $x < 7$
integers	the numbers in the set {… , -4, -3, -2, -1, 0, 1, 2, 3, 4, …}
inverse operation	an action that cancels a previously applied action *Example*: Subtraction is the inverse operation of addition.
length (l)	a one-dimensional measure that is the measurable property of line segments

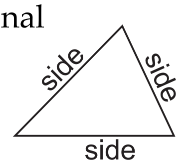
multiplicative identity	the number one (1), that is, multiplying by 1 does not change a number's value *Example*: 5 x 1 = 5
multiplicative inverses	any two numbers with a product of 1; also called *reciprocals* *Example*: 4 and $\frac{1}{4}$
natural numbers	the counting numbers {1, 2, 3, 4, …}
negative numbers	numbers less than zero
number line	a line on which numbers can be written or visualized
opposites	two numbers whose sum is zero *Example*: $\underbrace{-5 + 5}_{\text{opposites}} = 0$ *or* $\underbrace{\frac{2}{3} + -\frac{2}{3}}_{\text{opposites}} = 0$
order of operations	the order of performing computations in parentheses first, then exponents or powers, followed by multiplication and division, then addition and subtraction *Examples*: 5 + (12 – 2) ÷ 2 – 3 x 2 = 5 + 10 ÷ 2 – 3 x 2 = 5 + 5 – 6 = 10 – 6 = 4
origin	the graph of zero (0) on the number line *or* the intersection of the x-axis and the y-axis in a coordinate plane, described by the ordered pair (0, 0)
perfect square	a number whose square root is a whole number *Example*: 25 is a perfect square because 5 x 5 = 25

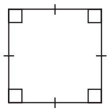
point	a location in space that has no length or width
positive numbers	numbers greater than zero
power (of a number)	an exponent; the number that tells how many times a number is used as a factor *Example*: In 2^3, 3 is the power.
prime factorization	writing a number as the product of prime numbers *Example*: $24 = 2 \times 2 \times 2 \times 3 = 2^3 \times 3$
prime number	any whole number with only two factors, 1 and itself *Example*: 2, 3, 7, 11, etc.
product	the result of a multiplication *Example*: In $6 \times 8 = 48$, 48 is the product.
quotient	the result of a division *Example*: In $42 \div 7 = 6$, 6 is the quotient.
radical	an expression that has a root (square root, cube root, etc.) *Example*: $\sqrt{25}$ is a radical Any root can be specified by an index number, *b*, in the form $\sqrt[b]{a}$ (e.g., $\sqrt[3]{8}$). A radical without an index number is understood to be a square root.



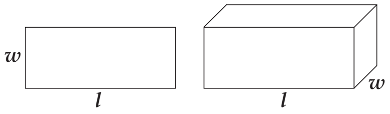
radical

radical sign ($\sqrt{\ }$)	the symbol ($\sqrt{\ }$) used before a number to show that the number is a *radicand*
radicand	a number that appears under a radical sign *Example*: In $\sqrt{25}$, 25 is the radicand.
reciprocals	two numbers whose product is 1; also called *multiplicative inverses* *Example*: Since $\frac{3}{4} \text{ x } \frac{4}{3} = 1$, the reciprocal of $\frac{3}{4}$ is $\frac{4}{3}$.
root	an equal factor of a number *Example*: In $\sqrt{144} = 12$, 12 is the square root. In $\sqrt[3]{125} = 5$, 5 is the cube root.
rounded number	a number approximated to a specified place *Example*: A commonly used rule to round a number is as follows. • If the digit in the first place after the specified place is 5 or more, *round up* by adding 1 to the digit in the specified place ($\overset{\curvearrowleft}{\underline{4}6}1$ rounded to the nearest hundred is 500). • If the digit in the first place after the specified place is less than 5, *round down* by *not* changing the digit in the specified place ($\overset{\curvearrowleft}{\underline{4}4}1$ rounded to the nearest hundred is 400).
side	the edge of a two-dimensional geometric figure *Example*: A triangle has three sides.



simplest form	a fraction whose numerator and denominator have no common factor greater than 1 *Example*: The simplest form of $\frac{3}{6}$ is $\frac{1}{2}$.
simplify a fraction	write fraction in lowest terms or simplest form
simplify an expression	to perform as many of the indicated operations as possible
solution	any value for a variable that makes an equation or inequality a true statement *Example*: In $y = 8 + 9$ $y = 17$ 17 is the solution.
solve	to find all numbers that make an equation or inequality true
square	a rectangle with four sides the same length 
square (of a number)	the result when a number is multiplied by itself or used as a factor twice *Example*: 25 is the square of 5.
square root (of a number)	one of two equal factors of a number *Example*: 7 is the square root of 49.
square units	units for measuring area; the measure of the amount of an area that covers a surface

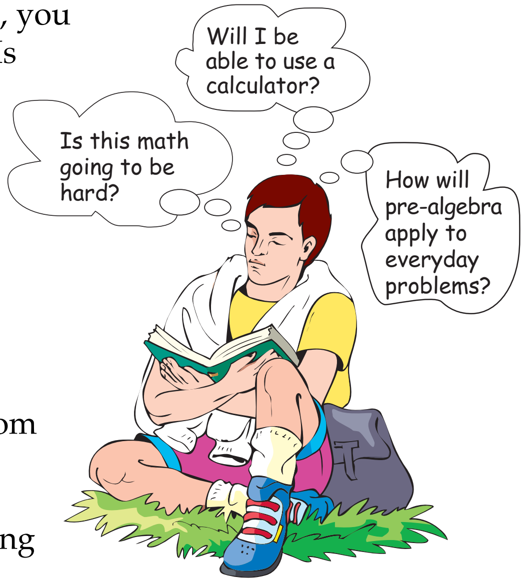
standard form	a method of writing the common symbol for a numeral *Example*: The standard numeral for five is 5.
substitute	to replace a variable with a numeral *Example*: $8(a) + 3$ $8(5) + 3$
sum	the result of an addition *Example*: In $6 + 8 = 14$, 14 is the sum.
unit (of length)	a precisely fixed quantity used to measure measurement in inches, feet, yards, and miles, or centimeters, meters, and kilometers
value (of a variable)	any of the numbers represented by the variable
variable	any symbol that could represent a number
whole number	any number in the set {0, 1, 2, 3, 4, …}
width (*w*)	a one-dimensional measure of something side to side



Unit 1: Number Sense, Concepts, and Operations

Introduction

As you get ready to study pre-algebra, you probably are thinking several things. Is this math going to be very different from what I've already had? Is it going to be hard? Will I be able to use a calculator? How will pre-algebra apply to everyday problems?



Pre-algebra is best described as a link between arithmetic and algebra.

- You will make the transition from arithmetic, which is mostly numerical, to problems that require more advanced reasoning skills.
- You will be working with *variables*, or symbols that represent numbers.
- You will be learning how to handle very large and very small numbers in an efficient fashion.
- You will have many opportunities to become proficient with a calculator.

Mastery of these skills is essential in our ever-changing technical world. Proficiency in pre-algebra will help you make the leap from numerical thinking to the more abstract thinking required in algebra and geometry.

Lesson One Purpose

- Associate verbal names, written word names, and standard numerals with integers, rational numbers, irrational numbers, and real numbers. (MA.A.1.4.1)
- Understand concrete and symbolic representations of real numbers in real-world situations. (MA.A.1.4.3)

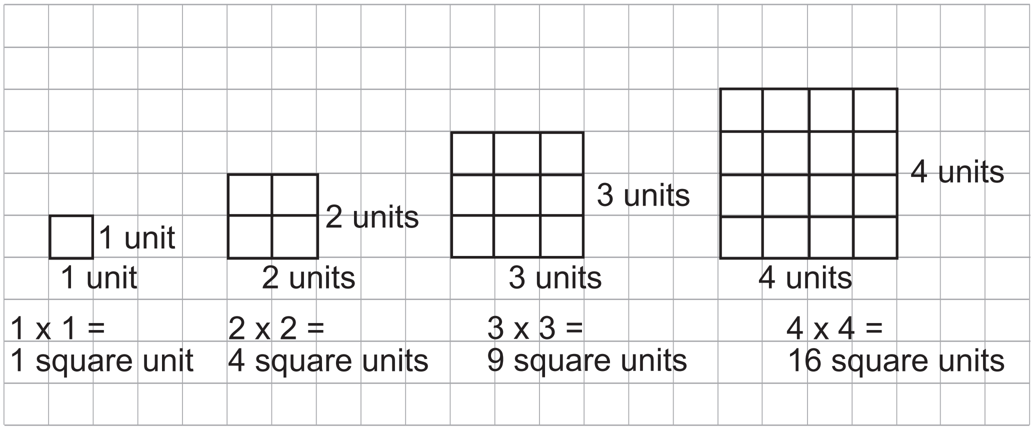
- Understand that numbers can be represented in a variety of equivalent forms, including integers, fractions, decimals, exponents, and radicals. (MA.A.1.4.4)
- Understand and use the real number system. (MA.A.2.4.2)
- Understand and explain the effects of addition, subtraction, multiplication, and division on real numbers, including square roots, exponents, and appropriate inverse relationships. (MA.A.3.4.1)
- Add, subtract, multiply, and divide real numbers, including square roots and exponents, using appropriate methods of computing, such as mental mathematics, paper and pencil, and calculator. (MA.A.3.4.3)

Writing Whole Numbers

There are many ways to write **whole numbers**. A *whole number* is any number in the set {0, 1, 2, 3, 4, …}. For example, to write a whole number, you may sometimes use an **exponent**, or the **power of a number**.

Using Exponents

Study the **squares** below. Each small *square* has four **sides** of the same **unit length (*l*)**.



One way to describe the **area (*A*)** of the squares above is by using *exponents*. The *area* is the inside region of a two-dimensional figure. The area is measured in **square units**.

The area of each square above can be described using exponents. An exponent tells how many times the **base** occurs as a **factor**. See below.

1 x 1 is $1 \bullet 1$ *or* 1^2
2 x 2 is $2 \bullet 2$ *or* 2^2
3 x 3 is $3 \bullet 3$ *or* 3^2
4 x 4 is $4 \bullet 4$ *or* 4^2

The numbers 1^2, 2^2, 3^2, and 4^2 on the previous page are numerical **expressions** in *exponential form*. Exponential form has two parts—the *base* and the *exponent*. For example, 2^5 is the exponential form of 2 x 2 x 2 x 2 x 2. The numeral two (2) is called the *base*, and the numeral five (5) is called the *exponent*.

The value is 32 because $2^5 = 2 \bullet 2 \bullet 2 \bullet 2 \bullet 2 = 32$.

Exponential form can also be defined as the *power of a number* written as an exponent. The exponent is the number that tells how many times a base is used as a *factor*. A factor divides evenly into another number.

For example:

32 can be written 2^5, or 2 to the fifth power.

How to Read Powers

These *powers* are read as follows:

6^2	six **squared** *or* six to the second power
10^3	ten **cubed** *or* ten to the third power
7^4	seven to the fourth power
1^{10}	one to the tenth power

Remember:
The meaning of 10^3 is $10 \bullet 10 \bullet 10$. Ten is used as a factor 3 times. The value is 1,000.

The meaning of 4^3 is $4 \bullet 4 \bullet 4$. Four is used as a factor 3 times. The value is 64.

Special Numbers—Perfect Squares

Numbers like 1, 4, 9, 16, 25, 36, 49, and 64 are called **perfect squares** because

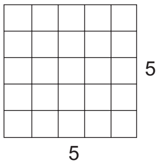
$1^2 = 1$
$2^2 = 4$
$3^2 = 9$
$4^2 = 16$
$5^2 = 25$
etc.

are the result of squaring a whole number.

The *square of a number* is the result when a number is used as a factor twice.

25 is the square of 5 because

$5^2 = 5 \text{ x } 5 = 25$



Here is another way to think of a *perfect square*. A perfect square is the **product** when a number is multiplied by itself.

25 is the square of 5 and a *perfect square* because it is the *product* of an **integer** {… , -2, -1, 0, 1, 2, …} multiplied by itself: 5 times 5. Here are other examples.

Perfect Squares

Number	1	2	3	4	5	6	7	8	← **natural numbers**
Square	1	4	9	16	25	36	49	64	← **perfect squares**

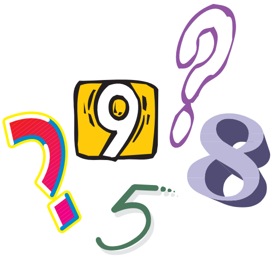
Numbers in the second row are called *perfect squares*. Perfect squares each have a square root that is a whole number.

Practice

Complete the following chart. The first one has been done for you.

Exponential Form	Meaning	Value	Verbal Description
1. 3^4	3 • 3 • 3 • 3	81	3 to the fourth power
2.	10 x 10 x 10		
3.		100	
4. 8^3			

Answer the following. Show all your work.



5. I'm thinking of a number.
   It is *greater than* 5^2.
   It is *less than* 15^2.
   Two and five *divide* my number.
   It is a *perfect square*.

   What is my number? ____________

6. What is the *largest* number that you can write using only 2 digits?

   Hint: Think about *powers* because it is *not* 99.

   ____

7. Wade the window washer hires himself out on the following basis: He charges

1 cent for the 1st window
2 cents for the 2nd window
4 cents for the 3rd window
8 cents for the 4th window
16 cents for the 5th window … and so forth

Window Washing Rates

Rate	Window
0.01	1st
0.02	2nd
0.04	3rd
0.08	4th
0.16	5th
.	.
.	.
.	.
?	32nd

At this rate, what would his cleaning bill be for the 32nd window?

____

Hint: Rewrite 2, 4, 8, and 16 in exponential form, with a base of 2.

$2^0 = 1$
$2^1 = 2$
$2^2 = 4$
$2^3 = 8$

Remember: Any number (except zero) to the zero power is defined as 1. So, $5^0 = 1$ and $64^0 = 1$.

Powers of 10

If you multiply 10s together, the product is called a *power of 10*. An exponent can be used to show a power of 10. The exponent tells the number of times that 10 is a factor.

10^2 has a value of 100

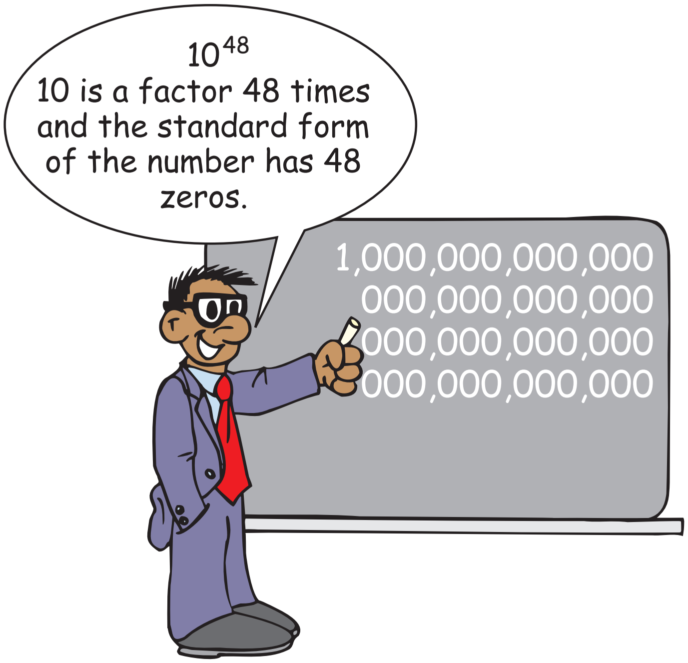
10^3 has a value of 1,000

10^4 has a value of 10,000

Remember: The **standard form** of a number is a method of writing the common symbol for a numeral. The *standard form* for eight is 8. When the base of a number is 10, the exponent and the number of zeros for the standard form are the same.

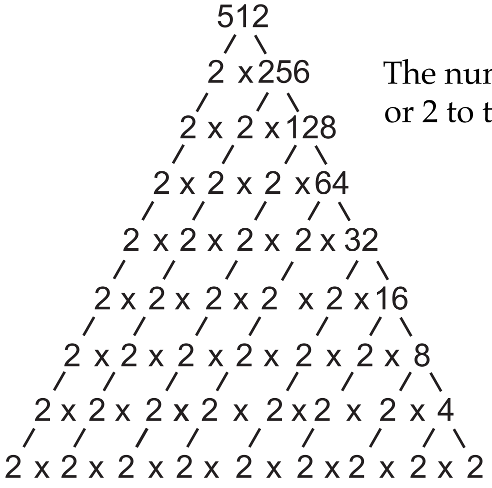
This is a nice shortcut. For example:

$$\textbf{base} \rightarrow \underbrace{10^{\overset{\textbf{exponent}}{\swarrow}4}}_{\textbf{exponential form}} = \underbrace{\underbrace{10 \bullet 10 \bullet 10 \bullet 10}_{\textbf{4 factors of 10}}}_{\textbf{factor form}} = \underbrace{\underbrace{10{,}000}_{\textbf{4 zeros to represent the exponent 4}}}_{\textbf{standard form}}$$



Exponential Form

What if you are asked to write 512 in *exponential form*? You would need to do a *factor tree* to find the answer.



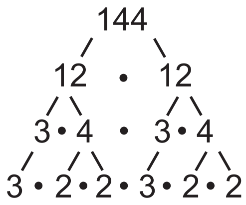
The number 512 can be written as 2^9, or 2 to the ninth power.

This is 2^9.

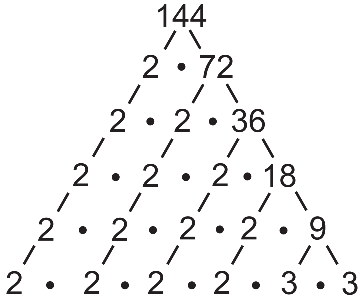
2^9 is the exponential form of 512.

Remember: A **prime number** is any whole number with only two factors, 1 and itself. 2 is a *prime number*. Don't stop until you have *all* prime numbers on the bottom of the factor tree.

Here is a second example done two ways. You can start with any factor of 144.



This is $2^4 \bullet 3^2$.



This is $2^4 \bullet 3^2$.

$2^4 \bullet 3^2$ is the exponential form of 144.

You may also try "upside down" dividing. In this method, begin dividing by the smallest *prime number* that is a factor of the original number. Continue the process until the last **quotient** or the result of the division problem below is 1.

For example:

$2\overline{)2{,}000}$	The smallest prime number factor of 2,000 is 2.
$2\overline{)1{,}000}$	The smallest prime number factor of 1,000 is 2.
$2\overline{)\ 500\ }$	The smallest prime number factor of 500 is 2.
$2\overline{)\ 250\ }$	The smallest prime number factor of 250 is 2.
$5\overline{)\ 125\ }$	The smallest prime number factor of 125 is 5.
$5\overline{)\ \ 25\ }$	The smallest prime number factor of 25 is 5.
$5\overline{)\ \ \ 5\ }$	The smallest prime number factor of 5 is 5.
1	

The **prime factorization** is the writing of a number as the product of prime numbers. The *prime factorization* of 2,000 includes each of these prime divisors: 2 x 2 x 2 x 2 x 5 x 5 x 5 = 2^4 x 5^3.

$2^4 \bullet 5^3$ is the exponential form of 2,000.

Practice

Answer the following. **Show all of your work.**

1. Write the following in *exponential form*. Use *factor trees* or "*upside down*" dividing.

   a. 81 = ________________________

   c. 56 = ________________________

   b. 84 = ________________________

2. Write the letter of the correct answer.

   Which *expression* represents $5^3 \bullet 10^4$? ____________

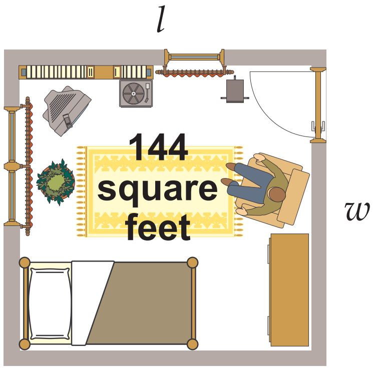
   a. 25 x 5 x 40

   b. 15 x 40

   c. 125 x 400

   d. 5 x 5 x 5 x 10 x 10 x 10 x 10

3. The area (A) of your bedroom is 144 square feet. If the room is a square, find the length (l) and **width (w)**.



area (A) = 144

length (l) = ____________

width (w) = ____________

4. Which is larger, 3^5 or 5^3? Explain why.

Answer: ____________

Explanation: ____

____

____

____

5. What is the *value* of 10^9? ______________________________

Practice

Match each definition with the correct term. Write the letter on the line provided.

______ 1. the number of times the base occurs as a factor

______ 2. a rectangle with four sides the same length

______ 3. the number that is used as a factor a given number of times

______ 4. a collection of numbers, symbols, and/or operation signs that stands for a number

______ 5. any number in the set {0, 1, 2, 3, 4, …}

______ 6. a one-dimensional measure that is the measurable property of line segments

______ 7. the inside region of a two-dimensional figure measured in square units

______ 8. the edge of a two-dimensional geometric figure

______ 9. a number or expression that divides exactly another number

______ 10. a precisely fixed quantity used to measure measurement in inches, feet, yards, and miles; centimeters, meters, and kilometers

______ 11. units for measuring area; the measure of the amount of an area that covers a surface

A. area (*A*)

B. base (of an exponent)

C. exponent

D. expression

E. factor

F. length (*l*)

G. side

H. square

I. square units

J. unit (of length)

K. whole number

Practice

Use the list below to write the correct term for each definition on the line provided.

cube (power)
integers
perfect square
power (of a number)
prime factorization
prime number
product
quotient
square (of a number)
standard form
width (*w*)

______________________________ 1. the numbers in the set {… , -4, -3, -2, -1, 0, 1, 2, 3, 4, …}

______________________________ 2. the result of a multiplication

______________________________ 3. writing a number as the product of prime numbers

______________________________ 4. the third power of a number

______________________________ 5. a method of writing the common symbol for a numeral

______________________________ 6. a number whose square root is a whole number

______________________________ 7. a one-dimensional measure of something side to side

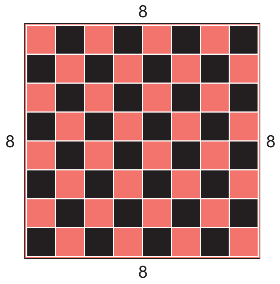
______________________________ 8. the result when a number is multiplied by itself or used as a factor twice

______________________________ 9. any whole number with only two factors, 1 and itself

_____________________________ 10. the result of a division

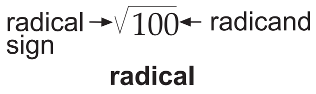
_____________________________ 11. an exponent; the number that tells how many times a number is used as a factor

Using Square Roots



A checkerboard is a perfect square containing 64 little squares. Each side of a square has the same length and width. We know that $8^2 = 64$, so each side of the checkerboard is 8 *units*. The opposite of squaring a number is called *finding the* **square root (of a number)**. The *square root* of 64 or $\sqrt{64}$ is 8.

The square root of a number is shown by the symbol $\sqrt{}$, which is called a **radical sign** or *square root sign*. The number underneath is called a **radicand**. The **radical** is an expression that has a **root**. A *root* is an equal factor of a number.



$$\sqrt{100} = 10 \text{ because } 10^2 = 100$$

$$\sqrt{9} = 3 \text{ because } 3^2 = 9$$

$$\sqrt{121} = 11 \text{ because } 11^2 = 121$$

All the numbers used in this lesson so far have been easy. They have all been *perfect squares*.

For example, 1, 4, 9, 16, 36, 49, 64, 81, and 100 are all perfect squares because their square roots are whole numbers. What do we do if the *radicand* is *not* a *perfect square*? We have three options to use to find the square root of a number:

Option 1: We can refer to a *square root chart*. Below is a partial table of squares and square roots. See Appendix A for a more complete table.

$\sqrt{6} \approx 2.449$

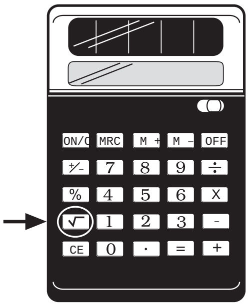
This answer is **rounded** to the nearest thousandth.

Remember: The symbol, ≈, means *is approximately equal to*. The symbol, ≈, is used with a number that describes another number without specifying it exactly.

Table of Squares and Approximate Square Roots

n	n^2	$\sqrt{n}$
1	1	1.000
2	4	1.414
3	9	1.732
4	16	2.000
5	25	2.236
6	36	2.449
7	49	2.646
8	64	2.828
9	81	3.000
10	100	3.162

Option 2: We can use a *calculator*. Look for a key with the $\sqrt{\ }$ symbol. Enter 6, hit this key, and you will get 2.44948974278. This result is a decimal *approximation* of the $\sqrt{6}$. You will have to *round* the number to the nearest thousandth.



$$\sqrt{6} = 2.44\underline{9}48974278$$
$$\sqrt{6} \approx 2.449$$

Option 3: We can **estimate**. We know

$$\sqrt{4} = 2$$

$$\sqrt{6} \approx ?$$

$$\sqrt{9} = 3$$

$\sqrt{6}$ is about half way between $\sqrt{4}$ and $\sqrt{9}$, so a good guess would be 2.5.

Note: Appendix B contains a list of mathematical symbols and their meanings and Appendix C contains formulas and conversions.

Practice

Use the **Table of Squares and Approximate Square Roots** *in* **Appendix A** *to complete the following.*

1. $\sqrt{33} \approx$ ____________
2. $\sqrt{80} \approx$ ____________
3. $\sqrt{49} =$ ____________

Answer the following.

4. List all the perfect squares between 100 and 400.

   Remember: 100 is a perfect square because $10^2 = 100$.

   ____

   ____

5. Find the square root of 30 using the following three options.

   table of square roots: ____

   calculator: ____

   estimation: ____

   Show all your work.

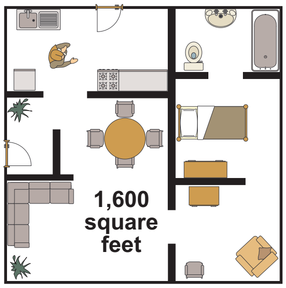
   ____

   ____

   ____

6. Your parents are planning to build a house with an area (A) of 1,600 square feet. A square house has more usable space than any other shape.

What would be the length (l) and width (w) of this house?



Remember: Area equals length times width, $A = lw$.

length = ____________

width = ____________

Find the perimeter (P) of this house.

Remember: Perimeter is the distance around the house. $P = 2l + 2w$

perimeter = ____

Circle the larger number.

7. 3^4 *or* $\sqrt{121}$

8. $\sqrt{169}$ *or* 14^2

9. $\sqrt{74}$ *or* 2^3

Write the letter of the correct answer.

10. Which of the following approximates $\sqrt{256.48}$? ____________

a. 128.24
b. 160.15
c. 512.96
d. 16.01

Use the **Table of Squares and Approximate Square Roots** *in* **Appendix A** *to complete the following.*

11. Using the table, run your finger down the column headed by n^2 until you find 1,024. We now know that $32^2 = 1{,}024$, so $\sqrt{1{,}024} = 32$. Use the table to find the following.

    a. $\sqrt{5{,}041}$ = ______________________________

    b. $\sqrt{3{,}249}$ = ______________________________

    c. $\sqrt{2{,}025}$ = ______________________________

    d. $\sqrt{9{,}801}$ = ______________________________

12. Using the table, guess what $\sqrt{6{,}800}$ equals. Explain your reasoning.

    ____

    ____

    ____

    ____

Check yourself: Use a calculator to check your answer.

Evaluating Algebraic Expressions—Order of Operations

Consider the following.

Evaluate $5 + 4 \bullet 3 =$

Is the answer 27 or is the answer 17? You could argue that both answers are valid, although 17 is the universally accepted answer. Mathematicians have agreed on the following **order of operations**.

Rules for Order of Operations

Always start on the *left* and move *to* the *right*.

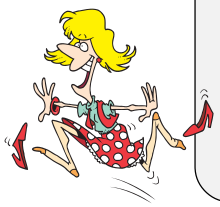
1.	Do operations inside *parentheses* first.	( ) , [ ], **or** $\frac{x}{y}$
2.	Then do all *powers* **or** *roots*.	x^2 **or** $\sqrt{x}$
3.	Next do *multiplication* **or** *division*, whichever comes first from left to right.	x **or** ÷
4.	Finally, do *addition* **or** *subtraction*, whichever comes first from left to right.	+ **or** –

Note: The fraction bar sometimes comes in handy to show grouping.

Example: $\frac{3x^2 + 8}{2} = (3x^2 + 8) \div 2$

The *order of operations* makes sure everyone doing the problem will get the same answer.

Some people remember these rules by using this mnemonic device to help their memory.



Please **P**ardon **M**y **D**ear **A**unt **S**ally*

Please	**P**arentheses
Pardon	**P**owers
My **D**ear	**M**ultiplication or **D**ivision
Aunt **S**ally	**A**ddition or **S**ubtraction

* Also known as **P**lease **E**xcuse **M**y **D**ear **A**unt **S**ally—**P**arentheses, **E**xponents, **M**ultiplication or **D**ivision, **A**ddition or **S**ubtraction.

Remember: You do multiplication **or** division—*whichever* comes first from *left to right*, and then addition **or** subtraction—*whichever* comes first from *left to right*.

Study the following.

$$25 - 3 \bullet 2 =$$

There are no parentheses. There are no powers or roots. We look for multiplication or division and find multiplication. We multiply. We look for addition or subtraction and find subtraction. We subtract.

$$\begin{aligned} 25 - 3 \bullet 2 &= \\ 25 - 6 &= \\ 19 & \end{aligned}$$

Study the following.

$$12 \div 3 + 6 \div 2 =$$

There are no parentheses. There are no powers or roots. We look for multiplication or division and find division. We divide. We look for addition or subtraction and find addition. We add.

$$\begin{aligned} 12 \div 3 + 6 \div 2 &= \\ 4 + 3 &= \\ 7 & \end{aligned}$$

If the rules were ignored, one might divide 12 by 3 and get 4, then add 4 and 6 to get 10, then divide 10 by 2 to get 5. Agreement is needed—using the agreed-upon *order of operations*.

Study the following.

$$30 - 3^3 =$$

There are no parentheses. We look for powers and roots and find powers, 3^3. We calculate this. We look for multiplication or division and find none. We look for addition or subtraction and find subtraction. We subtract.

$$30 - 3^3 =$$
$$30 - 27 =$$
$$3$$

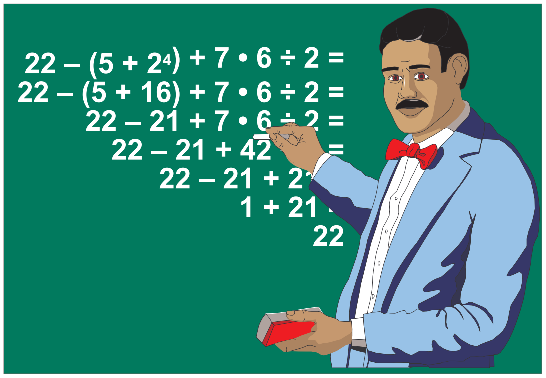
Please	**P**arentheses
Pardon	**P**owers
My	**M**ultiplication *or*
Dear	**D**ivision
Aunt	**A**ddition *or*
Sally	**S**ubtraction

Study the following.

$$22 - (5 + 2^4) + 7 \bullet 6 \div 2 =$$

We look for parentheses and find them. We must do what is inside the parentheses first. We find addition and a power. We do the power first and then the addition. There are no roots. We look for multiplication or division and find both. We do them in the order they occur, left to right, so the multiplication occurs first. We look for addition or subtraction and find both. We do them in the order they occur, left to right, so the subtraction occurs first.

$$22 - (5 + 2^4) + 7 \bullet 6 \div 2 =$$
$$22 - (5 + 16) + 7 \bullet 6 \div 2 =$$
$$22 - 21 + 7 \bullet 6 \div 2 =$$
$$22 - 21 + 42 \div 2 =$$
$$22 - 21 + 21 =$$
$$1 + 21 =$$
$$22$$



Practice

Answer the following.

Remember: *Symbols of inclusion—parentheses,* ( ), or *brackets,* [ ]—are used to tell which operation to do first.

1. $5 + 20 \div 4 =$

2. $5 \bullet 3^2 =$

3. $24 \div 2 \bullet 6 =$

4. $30 - 24 \div 4 \bullet 2 =$

5. $(6 + 2) \bullet 3 =$

6. $100 \div [(5 \bullet 2)^2] + 3 =$
Hint: First do what is inside the parenthesis, then do what is inside the brackets.

7. $\frac{19 - 5 \bullet (3 - 1)}{2 + 1} =$
Hint: Remember to simplify your answer.

8. $102 - 2 \bullet (3^4 - 51) =$

9. $10 - 2^3 \div 2 =$

Check yourself: *Use the list of* **scrambled answers** *below and check your answers to problems 1-9 above.*

3 4 6 10 18 24 42 45 72

10. Insert parentheses in the following so the answer is 10.

$17 + 3 \bullet 8 - 6 \div 4 = 10$

Use the list below to make each statement **true**. *These numbers may only be used* **once** *per statement.*

3 4 5 15

11. $\square + \square + \square = 12$

12. $\square \bullet \square \div \square = 20$

13. $\square \div \square \bullet \square = 25$

14. $\square + \square \bullet \square = 23$

15. $\square - \square \bullet \square = 3$

Answer the following.

16. $6 + 4 \bullet 5 - 2 =$

18. $6 + 4 \bullet (5 - 2) =$

17. $(6 + 4) \bullet (5 - 2) =$

19. $(6 + 4) \bullet 5 - 2 =$

Number the **order of operations** *in the correct* **order**. *Write the numbers on the line provided.*

______ 20. powers *or* roots

______ 21. multiplication *or* division

______ 22. addition *or* subtraction

______ 23. parentheses

Practice

Match each definition with the correct term. Write the letter on the line provided.

______ 1. the use of rounding and/or other strategies to determine a reasonably accurate approximation without calculating an exact answer

______ 2. the order of performing computations

______ 3. a number that appears under a radical sign

______ 4. the third power of a number

______ 5. the result of a multiplication

______ 6. the number of times the base occurs as a factor

______ 7. the symbol ($\sqrt{\ }$) used before a number to show that the number is a *radicand*

______ 8. the result when a number is multiplied by itself or used as a factor twice

______ 9. the inside region of a two-dimensional figure measured in square units

______ 10. a number approximated to a specified place

______ 11. an expression that has a root

______ 12. a number or expression that divides exactly another number

______ 13. an equal factor of a number

______ 14. one of two equal factors of a number

A. area

B. cube

C. estimation

D. exponent

E. factor

F. order of operations

G. product

H. radical

I. radical sign

J. radicand

K. root

L. rounded number

M. square (of a number)

N. square root (of a number)

Lesson Two Purpose

- Associate verbal names, written word names, and standard numerals with integers, rational numbers, irrational numbers, and real numbers. (MA.A.1.4.1)
- Understand and explain the effects of addition, subtraction, multiplication, and division on real numbers, including square roots, exponents, and appropriate inverse relationships. (MA.A.3.4.1)
- Add, subtract, multiply, and divide real numbers, including exponents, using appropriate methods of computing, such as mental mathematics, paper and pencil, and calculator. (MA.A.3.4.3)

Variables and Expressions

Suppose you are n years old today. In 4 years, your age can be described by the expression $n + 4$. Two years ago, your age would have been $n - 2$.

The letter n is a **variable**. A *variable* is any symbol that could represent a number. In this example, the variable represents your current age. Note that *any* letter of the alphabet or symbol can be used as a variable. A combination of operations, variables, and numbers is called a mathematical expression, algebraic expression, or simply an *expression*.

Here are sample phrases used to write mathematical expressions.

	Word Expression	**Mathematical Expression**
Addition:	5 **increased** by a number n	$5 + n$
	a number y plus 2	$y + 2$
	a number t increased by 4	$t + 4$
	the **sum** of a number b and 5	$b + 5$
	10 more than a number m	$m + 10$
Subtraction:	a number x minus 2	$x - 2$
	a number n less 3	$n - 3$
	5 less than a number t	$t - 5$
	a number t less than 5	$5 - t$
	a number c **decreased** by 2	$c - 2$
	the **difference** of a number x and 5	$x - 5$

	Word Expression	Mathematical Expression
Multiplication:	4 times a number y (form used most often is $4y$)	4 x y, $4(y)$, $4 \bullet y$, or $4y$
	the product of 3 and a number n	$3n$
	6 multiplied by a number t	$6t$
	twice a number p	$2p$
	$\frac{1}{2}$ a number y	$\frac{1}{2}y$
Division:	a number y divided by 2	$\frac{y}{2}$
	the *quotient* of t and 4	$\frac{t}{4}$
	a number c divided by 3	$\frac{c}{3}$
	3 divided by a number c	$\frac{3}{c}$
Power:	the square of x	x^2
	the cube of a	a^3
	the fourth power of x	x^4

Remember: $5n = 5$ x n, $5(n)$, $5 \bullet n$

$\frac{x}{3} = x \div 3$

Practice

Write a **mathematical expression** *for each* **word expression**.

1. 8 increased by a number y ______________________
2. 7 less than a number d ______________________
3. 15 decreased by s ______________________
4. 5 more than a number t ______________________
5. the sum of a number y and 4 ______________________
6. 12 less a number x ______________________
7. the product of 8 and a number d ______________________
8. 30 divided by a number b ______________________
9. the sum of a number r and 10 ______________________
10. a number t minus 6 ______________________
11. the quotient of 8 and a number c ______________________
12. 10 times a number y ______________________
13. twice a number q ______________________
14. the square of b ______________________
15. the cube of p ______________________

Read the following.

Study These Expressions

Words	Symbols
three times x plus y	$3x + y$
three times the sum of x and y	$3(x + y)$

In word expressions, look for key words that indicate that parentheses ( ) are to be used. Sometimes the words *sum*, *difference*, *quantity*, and *total* signal the use of parentheses.

Write a **mathematical expression** *for each* **word expression**. *Use* **parentheses ( )** *where appropriate.*

16. twice the sum of a number and 7 ____________________

17. one-half of the difference of a number x and 10 ____________________

18. twice a number increased by 8 ____________________

19. twice the total of a number and 5 ____________________

Answer the following.

20. One of the following does *not* belong. Write a sentence explaining why.

    a. Multiply 5 and a number then subtract 7.
    b. Subtract the product of 5 and a number from 7.
    c. $5x - 7$
    d. 7 less than the product of 5 and a number.

____

____

____

Evaluating Expressions

Here is how to evaluate mathematical expressions.

Suppose you are 16, and we let your age be represented by the variable *a*. The variable *a* now has a given **value** of 16. Calculate your age as follows:

a. in 4 more years	$a + 4 = \mathbf{16} + 4 = 20$
b. divided by 2	$\frac{a}{2} = \frac{\mathbf{16}}{2} = 8$
c. twice your age increased by 2	$2a + 2 = 2(\mathbf{16}) + 2 = 32 + 2 = 34$
d. the product of your age and 3	$3a = 3(\mathbf{16}) = 48$

Suppose you are 16, and we let your age be represented by the variable a.

Practice

Use the given **value** *of each* **variable** *below to evaluate each expression.*

$x = 6$	$y = 8$	$z = 2$

1. $x + 12 =$

2. $y - 5 =$

3. $x + z =$

4. $x - z =$

5. $y - x =$

6. $10 - y =$

7. $x + y - 2 =$

8. $20 - y - z =$

9. $x + x + x =$

10. $10 + x - x =$

Practice

Use the given **value** *of each* **variable** *below to evaluate each expression.*

$x = 6$ $y = 8$ $z = 2$

1. $5y =$

2. $\frac{y}{2} =$

3. $xz =$

4. $\frac{y}{z} =$

5. $5xy =$

6. $\frac{8}{y} =$

7. $5y + 10 =$

8. $\frac{20}{z} - x =$

9. $yz - x =$

10. $\frac{12}{x} + 12 =$

Practice

Use the given **value** *of each* **variable** *below to evaluate each expression.*

$r = 4$ **$s = 5$** **$t = 10$**

1. r increased by s
2. the sum of r and t
3. s less than t
4. t minus r
5. r more than 4
6. t divided by s
7. the product of r and s
8. s decreased by r
9. the sum of s and t decreased by 9
10. 12 divided by r, plus s
11. the cube of s increased by the sum of r and t
12. the square of s decreased by the quotient of t and s

Practice

Use the list below to write the correct term for each definition on the line provided.

decrease	**increase**	**variable**
difference	**sum**	

____________________ 1. any symbol that could represent a number

____________________ 2. the result of an addition

____________________ 3. to make greater

____________________ 4. the result of a subtraction

____________________ 5. to make less

Practice

Match each definition with the correct term. Write the letter on the line provided.

______ 1. the result of a multiplication

______ 2. the result of a division

______ 3. a collection of numbers, symbols, and/or operation signs that stands for a number

______ 4. an exponent; the number that tells how many times a number is used as a factor

______ 5. the result when a number is multiplied by itself or used as a factor twice

______ 6. the third power of a number

______ 7. any of the numbers represented by the variable

A. cube

B. expression

C. power (of a number)

D. product

E. quotient

F. square (of a number)

G. value

Lesson Three Purpose

- Understand and explain the effects of addition, subtraction, multiplication, and division on real numbers, including exponents and appropriate inverse relationships. (MA.A.3.4.1)
- Select and justify alternative strategies, such as using properties of numbers, including inverse, identity, and associative, that allow operational shortcuts for computational procedures in real-world or mathematical problems. (MA.A.3.4.2)

Solving Equations by Guessing

An **equation** is a mathematical sentence that *equates* one expression to another expression.

For example, you know:

$$2 + 2 = 4$$
$$2 \bullet 3^2 = 18$$

Now, consider this *equation*:

$$2(x + 3) = 14$$

What number could I use in place of the variable x, so that the left side is equal to the right side? We can guess. It must be a number that when multiplied by 2 equals 14.

$$2 \text{ x } ? = 14$$

We know that

$$2 \bullet 7 = 14,$$

and we know that

$$(4 + 3) = 7.$$

Therefore "4" is a **solution** to this equation.

4 is the *value* of the variable x.
$x = 4$

The equation is **solved** by **substituting** or *replacing x* in the original equation with the value of 4.

$$\begin{aligned} 2(x + 3) &= 14 \\ 2(4 + 3) &= 14 \\ 2(7) &= 14 \\ 14 &= 14 \end{aligned}$$

The *solution* of 4 makes the equation true.

Finding the value of a variable that makes a mathematical sentence true is called *solving the equation*. The value of the variable is called *the solution of the equation*.

Practice

Guess the answers *to the following.*

1. $x + 4 = 20$ (*Think*: What number can you add to 4 to get 20?)
   $x =$

2. $x - 4 = 20$ (*Think*: What number can you subtract 4 from to get 20?)
   $x =$

3. $4x = 20$ (*Think*: 4 times what number is 20?)
   $x =$

4. $\frac{x}{4} = 20$ (*Think*: What number can you divide by 4 to get 20?)
   $x =$

5. $6(x + 3) = 48$
   $x =$

6. $6x^2 = 24$
   $x =$

7. $(2x)^2 = 36$
   $x =$

Decide whether **5 is a solution** *to the following problems. Write* **yes** *if the solution is 5. Write* **no** *if the solution is* not *5.*

__________ 8. $23 - x = 6$

__________ 9. $2x + 3x = 25$

__________ 10. $\frac{130}{x} = 26$

Properties

Guessing is an acceptable way to solve simple equations, but we need to develop strategies which will help us solve harder equations. Before we do this, we need to examine some basic *properties* which will help us work with variables. These properties will help us make the leap from simple to more complex equations.

<table>
<tr><th colspan="2">Order (Commutative Property)</th></tr>
<tr><td>Commutative Property of Addition:
Numbers can be added in any order and the sum will be the same.
$10 + 2 = 2 + 10$
$x + 2 = 2 + x$</td><td>Commutative Property of Multiplication:
Numbers can be multiplied in any order and the product will be the same.
$2 \cdot 10 = 10 \cdot 2$
$2 \cdot x = x \cdot 2$</td></tr>
<tr><th colspan="2">Grouping (Associative Property)</th></tr>
<tr><td>Associative Property of Addition:
Numbers can be grouped in any order and the sum will be the same.
$(5 + 3) + 2 = 5 + (3 + 2)$
$(5 + x) + y = 5 + (x + y)$</td><td>Associative Property of Multiplication:
Numbers can be grouped in any order and the product will be the same.
$(5 \cdot 3) \cdot 2 = 5 \cdot (3 \cdot 2)$
$(5 \cdot x) \cdot y = 5 \cdot (x \cdot y)$</td></tr>
<tr><th colspan="2">Identity Properties</th></tr>
<tr><td>Additive Identity:
The sum of any number and zero is the number.
$5 + 0 = 5$
$x + 0 = x$</td><td>Multiplicative Identity:
The product of any number and one is the number.
$5 \cdot 1 = 5$
$x \cdot 1 = x$</td></tr>
<tr><th colspan="2">Inverse Properties</th></tr>
<tr><td>Additive Inverse:
The sum of any number and its additive inverse is 0.
$3 + -3 = 0$
3 and -3 are additive inverses, also called opposites.</td><td>Multiplicative Inverse:
The product of any number and its multiplicative inverse (reciprocal) is 1.
$4 \times \frac{1}{4} = 1$
4 and $\frac{1}{4}$ are multiplicative inverses, also called <i>reciprocals</i>.</td></tr>
</table>

Practice

Answer the following.

1. Why are division and subtraction not listed as commutative operations? Give examples to show your reasoning.

   Answer:____

   ____

   Examples:

2. Unfortunately, little Ben has lost his calculator and needs to add 4 + 5 + 16 + 15. Since we can add in any order, what would be a quick way to group the numbers and get the sum?

   Answer:____

   ____

3. Do the following calculation mentally by using the properties on the previous page, then explain your strategy.

   25 • 16 • 4

   Answer:____

   ____

   Explain: ____

   ____

   ____

   ____

Study the following examples on **simplifying** *before attempting the problems that follow. To* **simplify an expression***, perform as many of the indicated operations as possible. To* **simplify a fraction***, write the* **fraction** *in lowest terms or* **simplest form***.*

Example one

$(y + 2) + 3 =$

$y + (2 + 3) =$ associative property of addition

$y + 5$

Example two

$(7 \bullet x) \bullet 3 =$

$(x \bullet 7) \bullet 3 =$ commutative property of multiplication

$x \bullet (7 \bullet 3) =$ associative property of multiplication

$x \bullet 21 =$

$21 \bullet x =$ commutative property of multiplication

Simplify each expression *below using the* **properties** *from page 51.*

4. $(x + 3) + 4 =$

7. $11 + (2 + x) =$

5. $(x \bullet 6) \bullet 5 =$

8. $(m \bullet 4) \bullet 7 =$

6. $5 \bullet (4x) =$

9. $(1 \bullet x) \bullet 5 =$

Answer the following.

10. Is 0 + xy the same as yx? ____________

Explain: ____

____

____

____

11. Is b + 9a the same as 9a + b? ____________

Explain: ____

____

____

____

12. Is 2x – 5y the same as 5y – 2x? ____________

Explain: ____

____

____

____

13. Is 15 ÷ 3 the same as 3 ÷ 15? ____________

Explain: ____

____

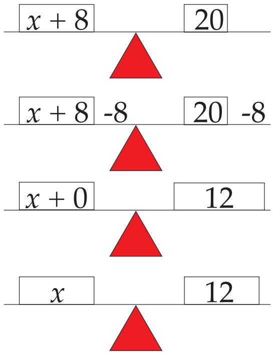
____

____

Solving One-Step Equations

Solving equations is very similar to keeping a see-saw balanced. For the see-saw to stay balanced, we know that we have to keep each side the same.

Whatever we do to one side of an equation, we have to do to the other.



subtract 8 from both sides

Remember: Our goal is to isolate the variable.

Study the following examples.

Left **Right**

a.
$$\begin{array}{rcr} \hline \multicolumn{1}{|r}{x+8} & = & \multicolumn{1}{r|}{20} \\ \hline -8 & & -8 \\ \hline x+0 & = & 12 \\ x & = & 12 \end{array}$$

subtract 8 from both sides

Addition and subtraction are **inverse** (or opposite) **operations**. We were able to *undo* the adding by subtracting because 8 and -8 are *additive inverses* or opposites.

b.
$$\begin{array}{rcr} \hline \multicolumn{1}{|r}{x-12} & = & \multicolumn{1}{r|}{13} \\ \hline +12 & & +12 \\ \hline x+\ \ 0 & = & 25 \\ x & = & 25 \end{array}$$

add 12 to both sides

We were able to undo the subtraction by adding.

c.
$$\begin{array}{rcr} \hline \multicolumn{1}{|r}{5x} & = & \multicolumn{1}{r|}{25} \\ \hline \frac{5x}{5} & = & \frac{25}{5} \\ {}^{1}\frac{\cancel{5}x}{{}_{1}\cancel{5}} & = & {}^{5}\frac{\cancel{25}}{{}_{1}\cancel{5}} \\ 1x & = & 5 \\ x & = & 5 \end{array}$$

divide both sides by 5

Multiplication and division are *inverse operations*. We can undo multiplication by dividing because 5 and $\frac{1}{5}$ are *multiplicative inverses*, or reciprocals.

d.
$$\begin{array}{rcl} \hline \multicolumn{1}{|r}{\frac{x}{5}} & = & \multicolumn{1}{l|}{10} \\ \hline \frac{5}{1} \bullet \frac{x}{5} & = & 5 \bullet 10 \\ {}^{1}\frac{\cancel{5}}{1} \bullet \frac{x}{{}_{1}\cancel{5}} & = & 5 \bullet 10 \\ 1 \bullet \frac{x}{1} & = & 5(10) \\ 1 \bullet x & = & 50 \\ 1x & = & 50 \\ x & = & 50 \end{array}$$

multiply both sides by 5

We can undo division by multiplying.

Remember: Multiplying a number by its *reciprocal* results in a product of 1. Any two numbers with a product of 1 are called *multiplicative inverses*. Here are some examples:

Number	Reciprocal	Product
$\frac{7}{8}$	$\frac{8}{7}$	$\frac{7}{8} \text{ x } \frac{8}{7} = 1$
8	$\frac{1}{8}$	$8 \text{ x } \frac{1}{8} = 1$
$\frac{1}{3}$	$\frac{3}{1}$	$\frac{1}{3} \text{ x } \frac{3}{1} = 1$

Left **Right**

e.

$$\begin{aligned} \boxed{\tfrac{1}{5}x &= 6} \\ 5 \bullet \tfrac{1}{5}x &= 5 \bullet 6 \\ {}^{1}\tfrac{\cancel{5}}{1} \bullet \tfrac{1}{{}_{1}\cancel{5}}x &= 5 \bullet 6 \\ 1 \bullet 1x &= 30 \\ 1x &= 30 \\ x &= 30 \end{aligned}$$

multiply both sides by 5

We can use the reciprocal of a *fraction* to isolate the variable.

f.

$$\begin{aligned} \boxed{\tfrac{3}{7}x &= 21} \\ \tfrac{7}{3} \bullet \tfrac{3}{7}x &= \tfrac{7}{3} \bullet 21 \\ \tfrac{{}^{1}\cancel{7}}{{}_{1}\cancel{3}} \bullet \tfrac{{}^{1}\cancel{3}}{{}_{1}\cancel{7}}x &= \tfrac{7}{{}_{1}\cancel{3}} \bullet \tfrac{{}^{7}\cancel{21}}{1} \\ 1 \bullet 1x &= 49 \\ 1x &= 49 \\ x &= 49 \end{aligned}$$

multiply both sides by $\frac{7}{3}$

Use the reciprocal of the fraction to isolate the variable.

Practice

Solve these equations. *Write each answer in* **simplest form**. **Show all steps**—*learning the* **procedure** *is as important as getting the* **solution**.

1. $5 + x = 14$

2. $\frac{1}{3}x = 51$

3. $3x = 51$

4. $x - 16 = 7$

5. $\frac{3}{4}x = 9$

6. $\frac{x}{4} = 2$

7. $6x = 2$

8. $2.5y = 37.5$

9. $n + 4.73 = 5.56$

Practice

Study the following examples. Then **solve the equations** *below. Write each answer in* **simplest form**. **Show all steps.**

Example 1

$$\begin{array}{rcll}
\multicolumn{3}{l}{\boxed{(x + 10) + 2 = \ 25}} & \\
x + (10 + 2) & = & 25 & \text{simplify by using associative property} \\
x + 12 & = & 25 & \\
-\,12 & & -\,12 & \text{subtract 12 from both sides} \\ \hline
x & = & 13 &
\end{array}$$

Example 2

$$\begin{array}{rcll}
\multicolumn{3}{l}{\boxed{10 \ = \ x \bullet 5}} & \\
x \bullet 5 & = & 10 & \text{flip the sides of the equation, if you wish} \\
\div\, 5 & & \div\, 5 & \text{divide both sides by 5} \\ \hline
x & = & 2 &
\end{array}$$

1. $7.2 = 0.36y$

2. $x \bullet 8 = 7$

3. $(3x) \bullet 4 = 24$

4. $(x + 2) + 9 = 12$

5. $(1 \bullet x) \bullet 5 = 20$

Practice

Match each definition with the correct term. Write the letter on the line provided.

_______ 1. two numbers whose sum is zero

_______ 2. a mathematical sentence that equates one expression to another expression

_______ 3. two numbers whose product is 1; also called *multiplicative inverses*

_______ 4. write fraction in lowest terms or simplest form

_______ 5. any value for a variable that makes an equation or inequality a true statement

_______ 6. the way in which three or more numbers are grouped for addition or multiplication does *not* change their sum or product
Example: (5 + 6) + 9 = 5 + (6 + 9) *or* (2 x 3) x 8 = 2 x (3 x 8)

_______ 7. the order in which any two numbers are added or multiplied does *not* change their sum or product
Example: 2 + 3 = 3 + 2 *or* 4 x 7 = 7 x 4

_______ 8. to replace a variable with a numeral

_______ 9. to perform as many of the indicated operations as possible

_______ 10. to find all numbers that make an equation or inequality true

A. associative property

B. commutative property

C. equation

D. opposites

E. reciprocals

F. simplify a fraction

G. simplify an expression

H. solution

I. solve

J. substitute

Practice

Use the list below to write the correct term for each definition on the line provided.

additive identity
additive inverses
fraction
inverse operation
multiplicative identity
multiplicative inverses
simplest form

____________________ 1. any number representing some part of a whole; of the form $\frac{a}{b}$

____________________ 2. the number zero (0), that is, adding 0 does not change a number's value

____________________ 3. a fraction whose numerator and denominator have no common factor greater than 1

____________________ 4. a number and its opposite whose sum is zero (0)

____________________ 5. the number one (1), that is, multiplying by 1 does not change a number's value

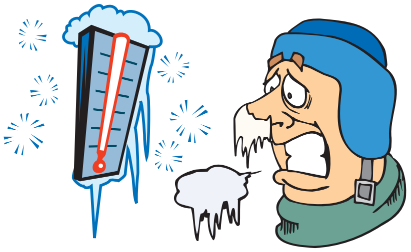
____________________ 6. any two numbers with a product of 1; also called *reciprocals*

____________________ 7. an action that cancels a previously applied action

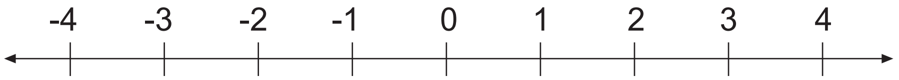
Lesson Four Purpose

- Understand that numbers can be represented in a variety of equivalent forms, including integers, fractions, decimals, exponents, and absolute value. (MA.A.1.4.4)
- Understand and use the real number system. (MA.A.2.4.2)
- Understand and explain the effects of addition, subtraction, multiplication, and division on real numbers, including exponents and appropriate inverse relationships. (MA.A.3.4.1)
- Add, subtract, multiply, and divide real numbers, including exponents, using appropriate methods of computing, such as mental mathematics, paper and pencil, and calculator. (MA.A.3.4.3)

Negative Numbers



Numbers were invented by people. The **positive numbers** 1, 2, 3, 4, 5, … were probably invented first, and were used for counting. Eventually, there was a need for numbers like 4.5 and $\frac{3}{4}$. *Positive numbers* are numbers greater than zero. **Negative numbers** were invented to represent things like temperatures below freezing, overdrawn bank balances, owing money, loss, or going backwards. *Negative numbers* are numbers less than zero. We frequently represent numbers on a **number line**.

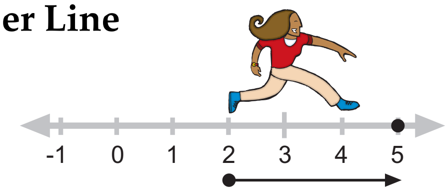


Numbers to the *left* of zero (also called the **origin** or the beginning) are *negative*, numbers to the *right* are *positive*. The set of numbers used on the number line above is called the *set of integers*.

This *set* can be written {… , -4, -3, -2, -1, 0, 1, 2, 3, 4, …}.

The bigger the number, the farther it is to the right. The smaller the number, the farther it is to the left. A number is considered positive if it does *not* have a sign written in front of it.

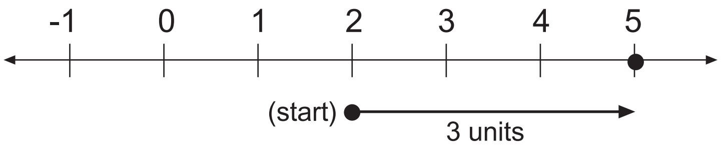
Adding Numbers by Using a Number Line



We are going to get a visual feel for adding integers by using a number line and taking "trips."

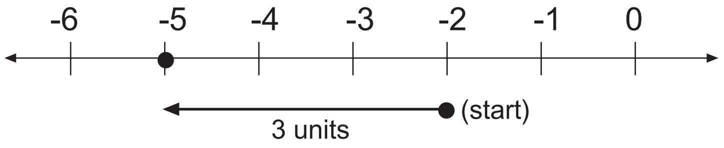
Trip One: Add 2 + 3

1. Start at 2.
2. Move 3 units to the right in the *positive* direction.
3. Finish at 5.
   So, 2 + 3 = 5.



Trip Two: Add -2 + -3

1. Start at -2.
2. Move 3 units to the left in the *negative* direction.
3. Finish at -5.
   So, -2 + -3 = -5.

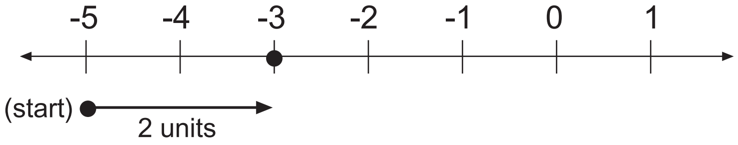


Trip Three: Add -5 + 2

1. Start at -5.

2. Move 2 units to the right in a *positive* direction.

3. Finish at -3.
   So, -5 + 2 = -3.

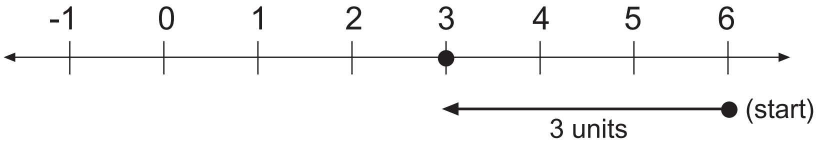


Trip Four: Add 6 + -3

1. Start at 6.

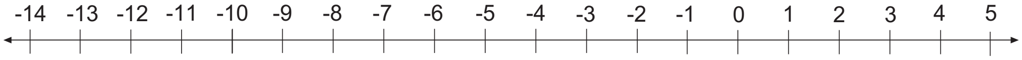
2. Move 3 units to the left in a *negative* direction.

3. Finish at 3.
   So 6 + -3 = 3.



Practice

Add *the following* **by visualizing moves** *on the* **number line**.



1. 6 + -5 = ____________
2. 7 + -3 = ____________
3. 5 + -2 = ____________
4. -4 + 1 = ____________
5. -5 + 3 = ____________
6. -1 + 4 = ____________
7. -6 + -5 = ____________
8. -12 + -1 = ____________
9. 12 + -1 = ____________
10. -5 + 5 = ____________

✓ **Check yourself: Add** *all your answers for problems 1-10 above. Did you get a* **sum of -7**? *If* **yes**, *complete the practice. If* **no**, *correct your work before continuing.*

Write **true** *if the statement is correct. Write* **false** *if the statement is* not *correct. If the statement is* **false**, *give an* **example** *to support your answer.*

__________ 11. The sum of two negative numbers is always *negative*.

____

__________ 12. A negative number plus a positive number is always *negative*.

____

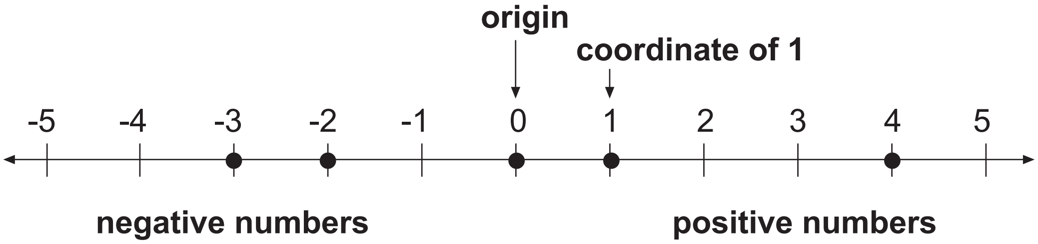
__________ 13. A positive number plus a negative number is always *positive*.

____

Graph of a Number

To **graph a number** means to draw a dot at the **point** that represents that *integer*. A *point* represents an exact location. The number paired with a point is called the **coordinate** of the point. The graph of zero (0) on a number line is called the *origin*.

Below is a graph of {-3, -2, 0, 1, 4}.



Number line shows coordinates of points -3, -2, 0, 1, and 4.

Practice

Complete the following.

1. Draw a number line and graph the integers 0, 4, -3, and 7.

Write the integer *that represents each* **situation described**. *Problem 2 has been done for you.*

2. 50 feet below sea level -50 feet
3. A loss of $5 ____________________
4. A gain of 10 pounds ____________________
5. 3 degrees below zero ____________________
6. An elevation of 1,000 feet ____________________
7. A profit of $200 ____________________

Order *the following integers from* **least to greatest**.

8. -1, 2, -2, 0, -4 ____________________
9. -7, -8, -9, 2, 3 ____________________
10. 6, 4, -4, 0, -5, 2 ____________________

Answer the following.

11. Is there a *greatest* positive integer? ____________________

    If so, name it. ______________________________

12. Is there a *least* positive integer? ____________________

    If so, name it. ______________________________

13. Is there an integer which is *neither* positive nor negative? ______

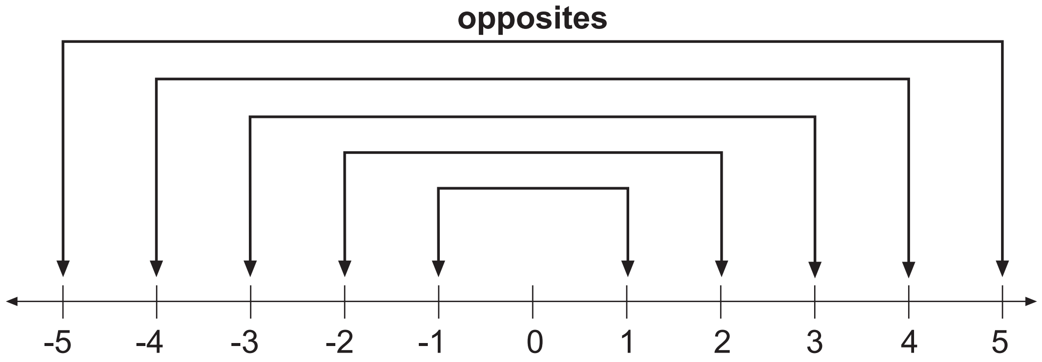
    If so, name it. ______________________________

Opposites and Absolute Value

We can visualize the process of adding by using a number line, but there are faster ways to add. To accomplish this, we must know two things: *opposites* or additive inverses and **absolute value**.

Opposites or Additives Inverses

5 and -5 are called *opposites*. Opposites are two numbers whose points on the number line are the same distance from 0 but in opposite directions.



The opposite of 0 is 0.

Every positive integer can be paired with a negative integer. These pairs are called *opposites*. For example, the opposite of 4 is -4 and the opposite of -5 is 5.

The opposite of 4 can be written -(4), so -(4) equals -4.

$$-(4) = -4$$

The opposite of -5 can be written -(-5), so -(-5) equals 5.

$$-(-5) = 5$$

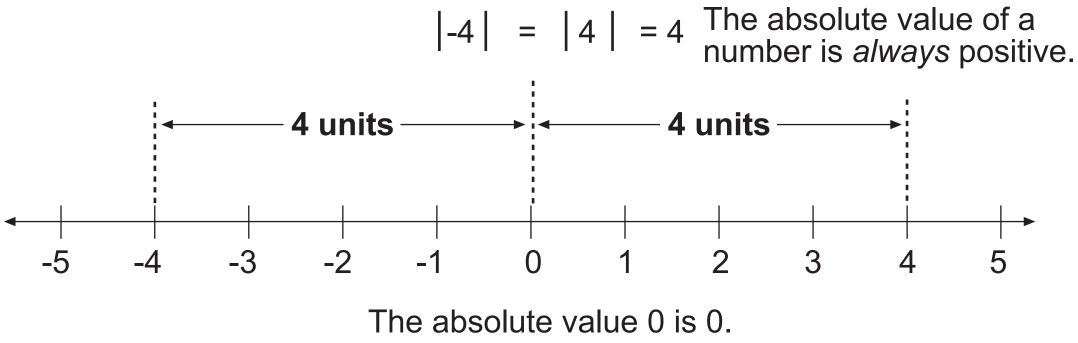
Two numbers are opposites or *additive inverses* of each other if their sum is zero.

For example: $4 + -4 = 0$
$-5 + 5 = 0$

Absolute Value

The *absolute value* of a number is the distance the number is from the *origin* or zero (0) on a number line. The symbol | | placed on either side of a number is used to show absolute value.

Look at the number line below. -4 and 4 are different numbers. However, they are the same distance in number of units from 0. Both have the same *absolute value* of 4. Absolute value is *always* positive because distance is always positive—you cannot go a negative distance.



$|-4|$ denotes the absolute value of -4.

$|-4| = 4$

$|4|$ denotes the absolute value of 4.

$|4| = 4$

The absolute value of 10 is 10. We can use this notation:

$$|10| = 10$$

The absolute value of -10 is also 10. We can use this notation:

$$|-10| = 10$$

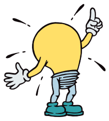
Both 10 and -10 are 10 units away from the origin. Consequently, the absolute value of both numbers is 10.

Now that we have this terminology under our belt, we can introduce two rules for adding numbers which will enable us to add quickly.

Adding Positive and Negative Integers

There are specific rules for adding positive and negative numbers.

1. **If the two integers have the *same sign*, *keep* the *sign* and *add* their absolute values.**



Example: -5 + -7
Think: Both signs are negative.

$|-5| = 5$

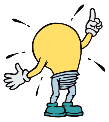
$|-7| = 7$

$5 + 7 = 12$

The sign will be negative because both signs were negative. Therefore, the answer is -12.

$-5 + -7 = -12$

2. **If the two integers have *opposite signs*, *subtract* the absolute values. The answer has the *sign* of the integer with the *greater* absolute value.**



Example: -8 + 3
Think: Signs are opposite.

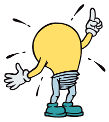
$|-8| = 8$

$|\ 3| = 3$

$8 - 3 = 5$

The sign will be negative because 8 has the greater absolute value. Therefore the answer is -5.

$-8 + 3 = -5$



Example: -6 + 8
Think: Signs are opposite.

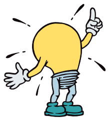
$|-6| = 6$

$|\ 8| = 8$

$8 - 6 = 2$

The sign will be positive because 8 has a greater absolute value. Therefore the answer is 2.

$-6 + 8 = 2$



Example: 5 + -7
Think: Signs are opposite.

$|\ 5\ | = 5$

$|-7| = 7$

$7 - 5 = 2$

The sign will be negative because 7 has the greater absolute value. Therefore the answer is -2.

$5 + -7 = -2$

Rules to Add Integers

- The sum of two positive integers is *positive*.
- The sum of two negative integers is *negative*.
- The sum of a positive integer and a negative integer takes the *sign of the greater absolute value*.
- The sum of a positive integer and a negative integer is zero if numbers have the *same absolute value*.

Practice

Answer the following.

1. 14 + -6 = ____________
2. 9 + -19 = ____________
3. -5 + -11 = ____________
4. -3 + 17 = ____________
5. -8 + 2 = ____________
6. -7 + -9 = ____________
7. 7 + -9 = ____________
8. -7 + 9 = ____________
9. -9 + 7 = ____________
10. -26 + -74 = ____________

11. $|-6| =$ ____________
12. -(-6) = ____________
13. -6 + 6 = ____________
14. -6 + -6 = ____________
15. -(-(-(-6))) = ____________
16. -6 + -3 + 14 = ____________
17. -10 + -7 + 2 = ____________
18. -30 + 6 + 29 = ____________
19. -17 + 6 + -11 + 22 =

    ____

Check yourself: *Use a* **calculator** *with a* $\boxed{+/-}$ **sign-change** *key to check problems 16-19 above.*

For example, for -16 + 4, you would enter 16 $\boxed{+/-}$ $\boxed{+}$ 4 $\boxed{=}$ and get the answer -12.

Practice

Use the given **value** *of each* **variable** *below to evaluate each expression.*

$a = -2$ $\quad b = 5$ $\quad c = -5$

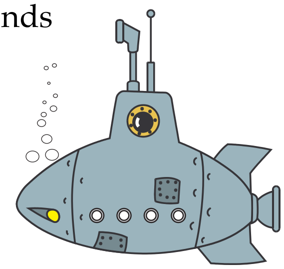
1. $8 + a =$
2. $13 + c =$
3. $a + b + c =$
4. $b + c + 10 =$
5. $|a| =$
6. $|a + b| =$

Answer the following.

Remember: Operations within **absolute value signs** must be done **first**.

7. $|-5| + |-7| =$
8. $|-5 + -7| =$
9. $-|-5 + -7| =$
10. $-|-5| + -|-7| =$
11. $-(-5) + -(-7) =$

12. A submarine is 1500 meters below sea level descends an additional 1200 meters. How far below sea level is the submarine now?



____

13. Below is a magic square. The sum of the numbers in each row, column, and diagonal of a magic square are the same. Each has a *sum* of 0. Complete the square.

1	**2**	**-3**
3		**-1**

Subtracting Integers

In the last section, we saw that 8 plus -3 equals 5.

$$8 + (-3) = 5$$

From elementary school we know that 8 minus 3 equals 5.

$$8 - 3 = 5$$

Below are similar examples.

$10 + (-7) = 3$ $\quad\quad$ $12 + (-4) = 8$

$10 - 7 = 3$ $\quad\quad$ $12 - 4 = 8$

These three examples show that there is a connection between adding and subtracting. As a matter of fact we can make any subtraction problem into an addition problem and vice versa.

This idea leads us to the following definition:

Definition of Subtraction
$\boldsymbol{a - b = a + (-b)}$

Examples: $8 - 10 = 8 + (-10) = -2$

$12 - 20 = 12 + (-20) = -8$

$-2 - 3 = -2 + (-3) = -5$

Even if we have $8 - (-8)$, this becomes

8 plus the opposite of -8, which equals 8.

$8 + -(-8) =$

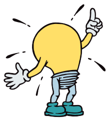
$8 + 8 = 16$

And -9 – (-3), this becomes

-9 plus the opposite of -3, which equals 3.

-9 + -(-3)

-9 + 3 = -6



Shortcut Two negatives become one positive!

10 – (-3) becomes 10 plus 3.

10 + 3 = 13

And -10 – (-3) becomes -10 plus 3.

-10 + 3 = -7

Generalization: Subtracting Integers

Subtracting an integer is the same as adding its opposite.

$$a - b = a + (-b)$$

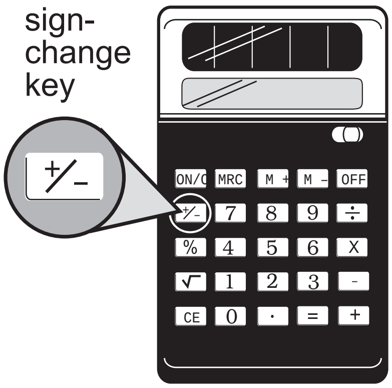
Practice

Answer the following.

1. 12 – 20 =
2. 15 – (-21) =
3. -30 – (-7) =
4. -40 – 30 =
5. -2 – (-8) =
6. 135 – (-165) =
7. 8 – 20 =
8. 0 – (-15) =
9. -12 – 10 =
10. -10 – (-5) =
11. 3 – 24 =
12. -50 – 20 =

Check yourself: *Use a* **calculator** *with a* **sign-change** *key to check problems 9-12.*

For example, for 18 – (-34), you would enter 18 $\boxed{-}$ 34 $\boxed{+/-}$ $\boxed{=}$ and get the answer 52.



Practice

Answer the following. The first one has been done for you.

Remember: Do operations within **parentheses first**.

1. 154 – (-43 – 27) =
   154 – (-43 + (-27)) =
   154 – (-70) =
   154 + 70 =
   224

2. 300 – (58 – 70) =

3. (33 – 45) – (64 – 78) =

4. 8 – (-3) – (4 – (-6)) =

5. 1 – 2 + 4 – 8 + 16 =

6. 200 – 195 + 71 – 66 + 18 – 29 =

7. $|2 - 9| =$

8. $|6 - (-4)| =$

9. $|9 - 6| + 5 =$

10. $|-14 + 3| - 8 =$

11. $-|-8 - (-5)| + 2 =$

12. A football team gained 3 yards, lost 5 yards, and gained 12 yards. What was the result?

____

13. A mountain climber is located at 4000 meters above sea level. A submarine is located at 1000 meters below sea level. What is their difference?

____

Multiplying Integers

What patterns do you notice?

3(4) = 12	3(-4) = -12
2 • 4 = 8	2 • -4 = -8
1(4) = 4	1(-4) = -4
0 • 4 = 0	0 • -4 = 0
-1(4) = -4	-1(-4) = 4
-2 • 4 = -8	-2 • -4 = 8
-3(4) = -12	-3(-4) = 12

Ask yourself:

- What is the sign of the product of two positive integers?
  3(4) = 12   2 • 4 = 8   *positive*
- What is the sign of the product of two negative integers?
  -1(-4) = 4   -2 • -4 = 8   *positive*
- What is the sign of the product of a positive integer and a negative integer or a negative integer and a positive integer?
  3(-4) = -12   -2 • 4 = -8   *negative*
- What is the sign of the product of any integer and 0?
  0 • 4 = 0   0 • -4 = 0   *neither*, zero is neither positive or negative

You can see that the sign of a *product* depends on the signs of the numbers being multiplied. Therefore, you can use the following rules to multiply integers.

Rules to Multiply Integers

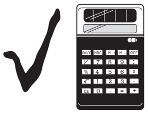
- The product of two positive integers is *positive*.
- The product of two negative integers is *positive*.
- The product of two integers with different signs is *negative*.
- The product of any integer and 0 is *0*.

Practice

Answer the following.

Alert: *For emphasis, the symbol* **($^{+}$)** *has been used for* **positive numbers**.

1. -5 • 0 = ______________
2. -6 • $^{+}$4 = ______________
3. $^{+}$9 • $^{+}$5 = ______________
4. -3 • -3 = ______________
5. $^{+}$7 • 0 = ______________
6. $^{+}$9 • -9 = ______________
7. $^{+}$4 • -7 = ______________
8. -9 • -8 = ______________
9. -3 • $^{+}$5 = ______________
10. -6 • -4 = ______________
11. -12 • $^{+}$12 = ______________
12. -15 • -10 = ______________
13. -16 • $^{+}$14 = ______________
14. $^{+}$12 • $^{+}$12 = ______________



Check yourself: *Use a* **calculator** *with a* $\boxed{+/-}$ **sign-change** *key to check problems 11-14 above.*

For example, for -13 • -7, you would enter 13 $\boxed{+/-}$ $\boxed{x}$ 7 $\boxed{+/-}$ $\boxed{=}$ and get the answer 91.

Practice

Use the given **value** *of each* **variable** *below to evaluate each expression.*

$a = -6$ $\quad$ $b = -3$ $\quad$ $c = 0$ $\quad$ $d = {}^{+}8$

1. $a + b =$

2. $c - b =$

3. $ad =$

4. $d + c =$

Check yourself: *Use the list of* **scrambled answers** *below and check your answers to problems 1-4 above.*

-48 $\quad$ -9 $\quad$ $^{+}3$ $\quad$ $^{+}8$

Did you get numbers 1-4 above correct? If **yes**, *complete the practice. If* **no**, *correct your work before continuing.*

5. $cb =$

6. $a + c =$

7. $d - b =$

8. $da =$

9. $a - b =$

10. $bb =$

11. $b + d =$

12. $d - a =$

13. $a(b + c) =$

14. $ab + ac =$

15. $a(b - c) =$

16. $ab - ac =$

17. $(a + b) + c =$

18. $a + (b + c) =$

19. $(a - b) - d =$

20. $a - (b - d) =$

Dividing Integers

Think:

1. What would you multiply ${}^+6$ by to get ${}^+42$?

   ${}^+6 \bullet ? = {}^+42$
   Answer: ${}^+7$ because ${}^+6 \bullet {}^+7 = {}^+42$

2. What would you multiply -6 by to get -54?

   $-6 \bullet ? = -54$
   Answer: ${}^+9$ because $-6 \bullet {}^+9 = -54$

3. What would you multiply -15 by to get 0?

   $-15 \bullet ? = 0$
   Answer: 0 because $-15 \bullet 0 = 0$

Remember: The result of dividing is a *quotient*.

Example:

42 divided by 7 results in a quotient of 6.

$$42 \div 7 = \underset{\underset{\text{quotient}}{\uparrow}}{6}$$

To find the quotient of 12 and 4 we write:

$$4\overline{)12} \quad or \quad 12 \div 4 \quad or \quad \frac{12}{4}$$

Each problem above is read "12 *divided by* 4." In each form, the quotient is 3.

$$\begin{array}{r} \text{quotient} \\ \downarrow \\ 3 \\ \text{divisor} \rightarrow 4\overline{)12} \\ \uparrow \\ \text{dividend} \end{array} \quad or \quad \underset{\underset{\text{dividend}}{\uparrow}}{12} \div \overset{\overset{\text{divisor}}{\downarrow}}{4} = \underset{\underset{\text{quotient}}{\uparrow}}{3} \quad or \quad \frac{\overset{\overset{\text{dividend}}{\downarrow}}{12}}{\underset{\underset{\text{divisor}}{\uparrow}}{4}} = 3 \leftarrow \text{quotient}$$

In $\frac{12}{4}$, the bar separating 12 and 4 is called a *fraction bar*. Just as *subtraction* is the *reverse of addition*, *division* is the *reverse of multiplication*. This means that *division* can be *checked by multiplication*.

$$\begin{array}{r}3\\4\overline{)12}\end{array} \quad \textit{because} \quad 3 \bullet 4 = 12$$

Division of integers is *related to* multiplication of integers. The sign rules for division can be discovered by writing a related multiplication problem.

For example,

same signs → positive quotient (or product)	**different signs → negative quotient (or product)**
$\frac{6}{2} = 3$ because $3 \bullet 2 = 6$	$\frac{-6}{2} = -3$ because $-3 \bullet 2 = -6$
$\frac{-6}{-2} = 3$ because $3 \bullet -2 = -6$	$\frac{6}{-2} = -3$ because $-3 \bullet -2 = 6$

Below are the rules used to divide integers.

Rules to Divide Integers

- The quotient of two positive integers is *positive*.
- The quotient of two negative integers is *positive*.
- The quotient of two integers with different signs is *negative*.
- The quotient of 0 divided by any nonzero integer is *0*.

Note the special division properties of 0:

$0 \div 9 = 0$ $\qquad$ $0 \div -9 = 0$

$\frac{0}{5} = 0$ $\qquad$ $\frac{0}{-5} = 0$

$\begin{array}{r}0\\15\overline{)\ 0}\end{array}$ $\qquad$ $\begin{array}{r}0\\-15\overline{)\ 0}\end{array}$

Remember: Division by 0 is *undefined*. The quotient of any number and 0 is not a number.

We say that $\frac{^{+}9}{0}$, $\frac{^{+}5}{0}$, $\frac{^{+}15}{0}$, $\frac{^{-}9}{0}$, $\frac{^{-}5}{0}$, and $\frac{^{-}15}{0}$ are *undefined*.

Likewise $\frac{0}{0}$ is undefined.

For example, try to divide 134 ÷ 0. To divide, think of the related multiplication problem.

$$? \times 0 = 134$$

Any number times 0 is 0—so mathematicians say that division by 0 is undefined.

Practice

Give each **quotient**.

1. $^{+}20 \div {}^{+}2 =$ ______________

4. $-36 \div {}^{+}9 =$ ______________

2. $^{+}24 \div -3 =$ ______________

5. $-56 \div -7 =$ ______________

3. $-30 \div -5 =$ ______________

✓ **Check yourself:** *Use the list of* **scrambled answers** *below and check your answers to problems 1-5 above.*

-8 -4 6 8 10

Alert: From this point on the raised plus sign ($^{+}$) will *not* be used when writing a positive integer.

6. $14 \div -2 =$ ______________

11. $\frac{36}{-9} =$ ______________

7. $-64 \div -8 =$ ______________

12. $\frac{49}{-7} =$ ______________

8. $0 \div 7 =$ ______________

13. $\frac{-36}{-6} =$ ______________

9. $28 \div -4 =$ ______________

14. $\frac{0}{10} =$ ______________

10. $-48 \div 8 =$ ______________

15. $\frac{-48}{-6} =$ ______________

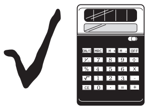
16. $\frac{7}{0}$ = ______________

17. $\frac{64}{8}$ = ______________

18. $\frac{-45}{9}$ = ______________

19. $\frac{0}{12}$ = ______________

20. $\frac{-72}{-9}$ = ______________



Check yourself: *Use a* **calculator** *with a* $\boxed{+/-}$ **sign-change** *key to check problems 16-20.*

For example, for $\frac{-54}{9}$, you would enter 54 $\boxed{+/-}$ $\boxed{\div}$ 9 $\boxed{=}$ and get the answer -6.

Practice

Answer the following.

1. $3 + 13 =$ ______________
2. $17 - 24 =$ ______________
3. $-9 \bullet 12 =$ ______________
4. $-10 \bullet -10 =$ ______________
5. $\frac{10}{-5} =$ ______________
6. $-45 + 20 =$ ______________
7. $48 - -30 =$ ______________
8. $16 + -28 =$ ______________
9. $3 \bullet -35 =$ ______________
10. $\frac{35}{-7} =$ ______________

Use the given **value** *of each* **variable** *below to* **evaluate each expression**.

$\boldsymbol{r} = -48$	$\boldsymbol{s} = 12$	$\boldsymbol{t} = -6$	$\boldsymbol{u} = 3$

11. $s + r =$

12. $t - r =$

13. $ut =$

14. $\frac{r}{t} =$

15. $\frac{s}{u} =$

16. $r - st =$

17. $\frac{r}{u} - s =$

18. $ts + tu =$

19. $u(t + s) =$

20. $u(s - t) =$

21. $s(t - t) =$

22. $\frac{r}{r} + \frac{u}{u} =$

23. $s^2 - \frac{s}{t} =$

24. $u^2 + t^2 =$

25. $r + s + t + u =$

Practice

Use the list below to complete the following statements.

absolute value
coordinate
graph a number
negative numbers
number line
origin
point
positive numbers

1. ____________________ are numbers greater than zero.

2. ____________________ are numbers less than zero.

3. We frequently represent numbers on a ____________________ .

4. To ____________________ means to draw a dot at the point that represents that integer.

5. A ____________________ represents an exact location.

6. The number paired with a point is called the ____________________ of the point.

7. The graph of zero (0) on a number line is called the ____________________ .

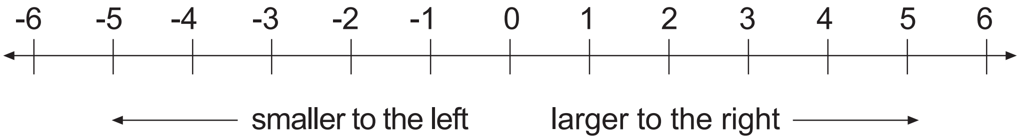
8. The ____________________ of a number is the distance the number is from the *origin* or zero (0) on a number line.

Lesson Five Purpose

- Understand the relative size of integers, rational numbers, and real numbers. (MA.A.1.4.2)
- Understand and use the real number system. (MA.A.2.4.2)
- Understand and explain the effects of addition, subtraction, multiplication, and division on real numbers, including appropriate inverse relationships. (MA.A.3.4.1)
- Use estimation strategies in complex situations to predict results and to check the reasonableness of results. (MA.A.4.4.1)

Comparing and Ordering Numbers

Here is how to use the number line to compare two *integers*.



Notice that values *increase* as you move to the *right*.

The number line above shows that

$-4 < -1$ $-1 < 2$ $0 < 3$ (< is less than)

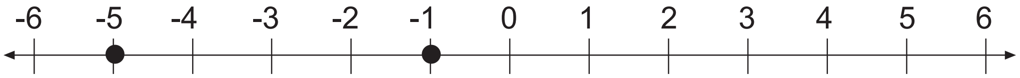
and it shows that

$3 > -4$ $1 > -2$ $0 > -4$ (> is greater than)

Each mathematical sentence above shows the *order* relationship between two quantities. The sentences compare two *unequal* expressions and use < (less than) and > (greater than) symbols.

These sentences state one expression is less than or greater than another expression and are called **inequalities**.

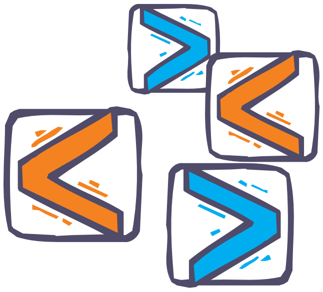
On the number line below, the integers graphed were used to write two *inequalities*.



Since -5 is to the left of -1, we can write -5 < -1.

Since -1 is to the right of -5, we can write -1 > -5.

Remember: The *is greater than* (>) or *is less than* (<) symbols always point to the lesser number.



Practice

Use the **numbers** *in each* **word sentence** *to write an* **inequality**. *Use* **< (is less than)** *or* **> (is greater than)** *to make a* **true** *statement.*

1. -5 is less than 10. ______________________________
2. 12 is greater than 8. ______________________________
3. -12° C is colder than -5° C. ________________________
4. 70 mph is faster than 50 mph. ______________________
5. $29.95 is more than $20.00. ______________________
6. 5 feet is taller than 3 feet. ________________________

Write <, >, *or* = *in each blank to make a* **true** *statement.*

7. 3 ______ 6
8. 2 ______ -5
9. -4 ______ 2
10. -5 ______ 6
11. 4 ______ -2
12. -5 ______ -3
13. -3 ______ 0
14. -8 ______ 5
15. 0 ______ 7
16. -3 ______ -12
17. -17 ______ 19
18. 17 ______ -19
19. 17 ______ 19
20. -17 ______ -19
21. |-3| ______ 3
22. -3 ______ |-3|
23. 6 ______ -6
24. |-5| ______ -5

Practice

Order the numbers from **least to greatest***.*

1. {7, 0, -3} ______________________________

2. {10, -9, -1} ______________________________

3. {0, -4, -8, -3} ______________________________

Use the list below to complete the following statements.

greater	**less**	**positive**
greater than	**negative**	**zero**

4. All ____________________ integers are greater than 0.

5. All ____________________ integers are less than 0.

6. ____________________ is neither positive or negative

7. Zero is ____________________ than any negative integer.

8. A negative integer is ____________________ than any positive integer.

9. A positive integer is ____________________ any negative integer.

Circle the letter of the correct answer.

10. If -3 is multiplied by a number greater than 1, which of the following describes the results?

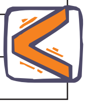
    a. A number between -3 and 3.
    b. A number greater than 3.
    c. A number less than -3.
    d. A number greater than -3.

Solving Inequalities

An inequality is formed when an *inequality symbol* is placed between two expressions. Solving an inequality is similar to solving an equation.

An inequality is any mathematical sentence that compares two unequal expressions using one of the following *inequality symbols*:

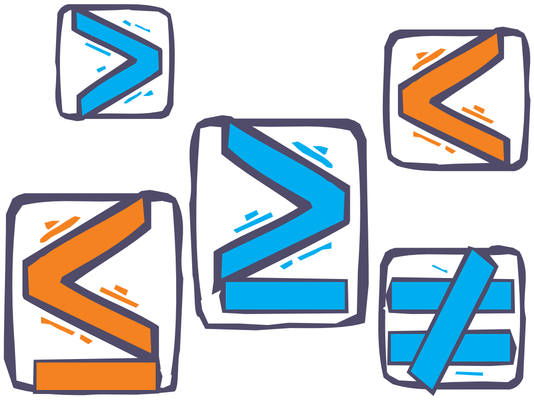
Inequality Symbol	**Meaning**
>	is greater than
≥	is greater than or equal to
<	is less than
≤	is less than or equal to
≠	is not equal to



In inequalities, numbers or expressions are compared.

These are inequalities.

$2 > 9 - x$	$2 + 2 \geq 4$	$y - 3 < 4$	$8 \text{ x } 4 \leq 50$	$6r \neq 8$
↑	↑	↑	↑	↑
is greater than	is greater than or equal to	is less than	is less than or equal to	is not equal to



Examples:

A. $x - 3 < 7$
$x - 3 \boxed{+ 3} < 7 \boxed{+ 3}$ add 3 to both sides
$x < 10$ simplify

B. $y + 4 > 12$
$y + 4 \boxed{- 4} > 12 \boxed{- 4}$ subtract 4 from both sides
$y > 8$ simplify

C. $\frac{d}{3} \geq 5$
$(3)\frac{d}{3} \geq 5(3)$ multiply both sides by 3
$(\cancel{3})\frac{d}{\cancel{3}} \geq 5(3)$ 3s cancel on the left side
$d \geq 15$ simplify

D. $2n \leq 14$
$\frac{2n}{2} \leq \frac{14}{2}$ divide both sides by 2
$\frac{\cancel{2}n}{\cancel{2}} \leq \frac{14}{2}$ 2s cancel on left side
$n \leq 7$ simplify

E. $15 > y - 2$ notice that the variable is on the right
$15 \boxed{+ 2} > y - 2 \boxed{+ 2}$ add 2 to both sides
$17 > y$ simplify

This *solution* may also be rewritten with the variable stated first. So instead of $17 > y$, it reads $y < 17$.

Remember: When the variable and the solution switch sides, the inequality symbol also changes to keep the solution the same.

As $y < 17$, the solution states that any number less than 17 is a solution of the problem. In other words, three solutions could be as follows:

-10, or 0, or 12.

However, keep in mind that there are infinitely many more numbers that are solutions.

Practice

State **two solutions** *for each* **inequality**.

1. $x < 5$ ______________
2. $y \geq 12.5$ ______________
3. $30 < x$ ______________
4. $x \leq 10$ ______________
5. $y > 0$ ______________

Solve each inequality. *Show all steps.*

6. $a + 3 < 11$
7. $y - 2 > 9$
8. $5t \leq 40$
9. $14 \leq t + 14$
10. $12 \geq s - 5$
11. $13n < 52$
12. $y - 14 \leq 16$
13. $\frac{x}{6} \geq 4$
14. $\frac{y}{2} \leq 100$
15. $\frac{a}{1.5} \leq 1.5$

Write an **inequality** *that* **represents the word sentence**. *Then* **solve** *the* **inequality**.

16. Ten plus *c* is greater than or equal to nineteen.

17. The difference of *y* and 7 is less than 23.

18. Two times *x* is less than twenty-two.

19. The product of *x* and 2 is greater than 40.

20. Thirteen is less than or equal to the quotient of *y* and 5.

Practice

Use the list below to complete the following statements.

cube (power)
decrease
exponent
factor
increase
inequality
product
quotient
square (of a number)
sum

____________________ 1. the number of times the base occurs as a factor

____________________ 2. the result of a multiplication

____________________ 3. the result of a division

____________________ 4. a sentence that states one expression is greater than (>), greater than or equal to (≥), less than (<), less than or equal to (≤), or not equal to (≠) another expression

____________________ 5. to make greater

____________________ 6. the result when a number is multiplied by itself or used as a factor twice

____________________ 7. the third power of a number

____________________ 8. the result of an addition

____________________ 9. to make less

____________________ 10. a number or expression that divides exactly another number

Practice

Match each definition with the correct term. Write the letter on the line provided.

______ 1. an exponent; the number that tells how many times a number is used as a factor

______ 2. the same number expressed in different forms

______ 3. any of the numbers represented by the variable

______ 4. a number's distance from zero (0) on the number line

______ 5. numbers less than zero

______ 6. numbers greater than zero

______ 7. the inside region of a two-dimensional figure measured in square units

______ 8. the order of performing computations in parentheses first, then exponents or powers, followed by multiplication and division, then addition and subtraction

______ 9. one of two equal factors of a number

______ 10. a fraction whose numerator and denominator have no common factor greater than 1

A. absolute value

B. area (*A*)

C. equivalent (form of a number)

D. negative numbers

E. order of operations

F. positive numbers

G. power (of a number)

H. simplest form

I. square root (of a number)

J. value (of a variable)

Unit Review

Part A

Match each **word expression** *with its* **equivalent mathematical expression***.*
Write the letter on the line provided.

Note: *Equivalent* means the same number expressed in different forms.

______ 1. the cube of x divided by 10, plus y

______ 2. 15 less a number y

______ 3. the square of x

______ 4. the sum of a number x and 15

______ 5. the quotient of 3 and a number x

______ 6. the product of 3 and a number x

______ 7. 15 less than a number y

______ 8. the cube of x increased by the sum of y and 10

______ 9. the sum of x cubed and y divided by 10

______ 10. a number x decreased by 3

A. $x + 15$

B. $\frac{3}{x}$

C. $x^3 + (y + 10)$

D. $y - 15$

E. $\frac{x^3}{10} + y$

F. $15 - y$

G. $\frac{x^3 + y}{10}$

H. x^2

I. $x - 3$

J. $3 \bullet x$

Complete the following chart.

Exponential Form	Meaning	Value	Verbal Description
11. 3^5			
12.	4 • 4 • 4		
13.		121	
14.			5 to the third power

15. Use a *factor tree* to write 36 in *exponential form*.

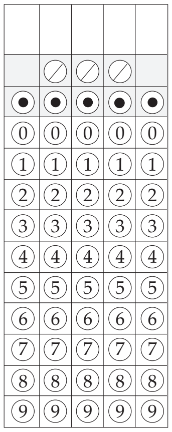
16. What is the *value* of 10^9? ____________________________________

Use either a **calculator** *or the* **Table of Squares and Approximate Square Roots** *in* **Appendix A** *to answer the following.* **Round to the nearest thousandth.**

Numbers 17-19 are **gridded-response items**.
Write answers along the top of the grid and correctly mark them below.

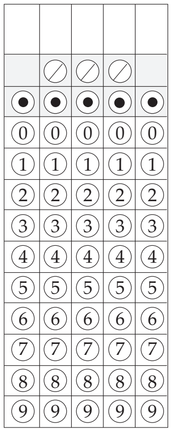
17. $\sqrt{89} \approx$

Mark your answer on the grid to the right.



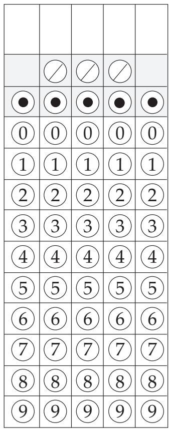
18. $\sqrt{1{,}600} =$

Mark your answer on the grid to the right.



19. $\sqrt{47} \approx$

Mark your answer on the grid to the right.



20. The area of a room is 289 square feet. If the room is a square, find the *length* and *width*.

length (l) = ____________ width (w) = ____________

Use the rules for the **order of operations** *to find the answer.*

21. $10 + 2^3 \div 2 =$

22. $4 \bullet 5^2 =$

23. $\frac{25 - 6(5 - 2)}{4 + 3} =$

24. $100 \div 10 \bullet 2 =$

25. $10 + 100 \div 2 =$

Unit Review

Part B

Simplify *the following expression.* **State the properties** *that you use.*

26. $(y \bullet 6) \bullet 4 =$

Use the given **value** *of each* **variable** *below to* **evaluate each expression**.

$\mathbf{x = 10}$	$\mathbf{y = 12}$	$\mathbf{z = 2}$

27. $\frac{xy}{z} - 5y =$

28. $2y + 10 =$

29. the square of x increased by the *sum* of y and z

30. the square of x decreased by the *quotient* of y and z

Do **not** *use your* **calculator** *to answer the following.*

31. $|-10| =$ ____________

32. -(-10) ____________

33. 10 + -20 = ____________

34. -10 + -20 = ____________

35. -10 – 20 = ____________

36. -10 – (-20) = ____________

37. (-10) • (-5) = ____________

Use the given **value** *of each* **variable** *to* **evaluate each expression**.

$\boldsymbol{a = 10}$ $\boldsymbol{b = -5}$ $\boldsymbol{c = -4}$ $\boldsymbol{d = 0}$

38. $a(b + c) =$

39. $ab - c =$

40. $bd + ba =$

Solve *the following. Write each answer in* **simplest form**. **Show all steps.**

41. $x + 8 = 24$

44. $\frac{x}{8} = 24$

42. $8x = 24$

45. $\frac{3}{4}x = 24$

43. $x - 8 = 24$

46. $(3x) \bullet 4 = 6$

Write in each blank <, or >, or = to make a true statement.

47. $|4|$ ______ $-(-4)$

48. -5 ______ -13

49. $|3 - 5|$ ______ $(-7) \bullet (-3)$

50. State **four** possible solutions of the **inequality** below.

$y + 4 \leq 10$

____

Unit 2: Measurement

This unit emphasizes the relationships between various types of numbers.

Unit Focus

Number Sense, Concepts, and Operations

- Understand the relative size of integers, rational numbers, and real numbers. (MA.A.1.4.2)
- Understand concrete and symbolic representations of real numbers in real-world situations. (MA.A.1.4.3)
- Understand that numbers can be represented in a variety of equivalent forms, including integers, fractions, decimals, percents, scientific notation, and exponents. (MA.A.1.4.4)
- Understand and use the real number system. (MA.A.2.4.2)
- Understand and explain the effects of addition, subtraction, multiplication, and division on real numbers, including exponents, and appropriate inverse relationships. (MA.A.3.4.1)
- Select and justify alternative strategies, such as using properties of numbers, including inverse, that allow operational shortcuts for computational procedures in real-world or mathematical problems. (MA.A.3.4.2)
- Add, subtract, multiply, and divide real numbers, including exponents, using appropriate methods of computing, such as mental mathematics, paper and pencil, and calculator. (MA.A.3.4.3)
- Use estimation strategies in complex situations to predict results and to check the reasonableness of results. (MA.A.4.4.1)

Measurement

- Use concrete and graphic models to derive formulas for finding rate, distance, and time. (MA.B.1.4.2)
- Relate the concepts of measurement to similarity and proportionality in real-world situations. (MA.B.1.4.3)
- Solve real-world problems involving rated measures (miles per hour, feet per second). (MA.B.2.4.2)

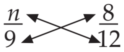
Vocabulary

Use the vocabulary words and definitions below as a reference for this unit.

additive inverses	a number and its opposite whose sum is zero (0); also called *opposites* *Example*: In the equation 3 + -3 = 0, 3 and -3 are additive inverses, or *opposites*, of each other.
canceling	dividing a numerator and a denominator by a common factor to write a fraction in lowest terms or before multiplying fractions *Example*: $\frac{15}{24} = \frac{{}^{1}\cancel{3} \bullet 5}{2 \bullet 2 \bullet 2 \bullet \cancel{3}_{1}} = \frac{5}{8}$
common denominator	a common multiple of two or more denominators *Example*: A common denominator for $\frac{1}{4}$ and $\frac{5}{6}$ is 12.
common factor	a number that is a factor of two or more numbers *Example*: 2 is a common factor of 6 and 12.

cross multiplication .... a method for solving and checking proportions; a method for finding a missing numerator or denominator in equivalent fractions or ratios by making the cross products equal

Example: To solve this proportion:



$$12 \text{ x } n = 9 \text{ x } 8$$
$$12n = 72$$
$$n = \frac{72}{12}$$
$$n = 6$$

Solution:

$$\frac{6}{9} = \frac{8}{12}$$

cross product .... the product of one numerator and the opposite denominator in a pair of fractions

Example:

Is $\frac{2}{5}$ equal to $\frac{6}{15}$?

$$\frac{2}{5} \overset{?}{=} \frac{6}{15}$$

$2 \text{ x } 15 \overset{?}{=} 5 \text{ x } 6$ The cross products are 2 x 15 and 5 x 6.

$30 = 30$ Both cross products equal 30. The cross products of equivalent fractions are equal.

Yes, $\frac{2}{5} = \frac{6}{15}$.

decimal number .... any number written with a decimal point in the number

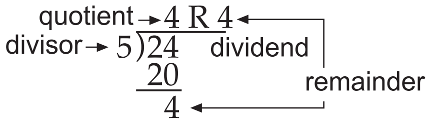
Example: A decimal number falls between two whole numbers, such as 1.5 falls between 1 and 2. Decimal numbers smaller than 1 are sometimes called decimal fractions, such as five-tenths is written 0.5.

decimal point	the dot dividing a decimal number's whole part from its fractional part
denominator	the bottom number of a fraction, indicating the number of equal parts a whole was divided into *Example*: In the fraction $\frac{2}{3}$ the denominator is 3, meaning the whole was divided into 3 equal parts.
difference	the result of a subtraction *Example*: In $16 - 9 = 7$, 7 is the difference.
digit	any one of the 10 symbols 0, 1, 2, 3, 4, 5, 6, 7, 8, or 9
divisor	a number by which another number, the dividend, is divided *Example*: In $7\overline{)42}$, $42 \div 7$, $\frac{42}{7}$, 7 is the divisor.
equation	a mathematical sentence that equates one expression to another expression *Example*: $2x = 10$
equivalent (forms of a number)	the same number expressed in different forms *Example*: $\frac{3}{4}$, 0.75, and 75%
estimation	the use of rounding and/or other strategies to determine a reasonably accurate approximation without calculating an exact answer

exponent (exponential form)	the number of times the base occurs as a factor *Example*: 2^3 is the exponential form of 2 x 2 x 2. The numeral two (2) is called the *base*, and the numeral three (3) is called the *exponent*.
expression	a collection of numbers, symbols, and/or operation signs that stands for a number *Example*: $4r^2$; $3x + 2y$; $\sqrt{25}$ Expressions do *not* contain equality (=) or inequality ($<$, $>$, $\leq$, $\geq$, or $\neq$) symbols.
factor	a number or expression that divides exactly another number *Example*: 1, 2, 4, 5, 10, and 20 are factors of 20.
fraction	any number representing some part of a whole; of the form $\frac{a}{b}$ *Example*: One-half written in fractional form is $\frac{1}{2}$.
greatest common factor (GCF)	the largest of the common factors of two or more numbers *Example*: For 6 and 8, 2 is the greatest common factor.
improper fraction	a fraction that has a numerator greater than or equal to the denominator *Example*: $\frac{5}{4}$ or $\frac{3}{3}$ are improper fractions.

integers	the numbers in the set {… , -4, -3, -2, -1, 0, 1, 2, 3, 4, …}
interest	the amount of money paid for the use of money
least common denominator (LCD)	the smallest common multiple of the denominators of two or more fractions *Example*: For $\frac{3}{4}$ and $\frac{1}{6}$, 12 is the least common denominator.
mixed number	a number that consists of both a whole number and a fraction *Example*: $1\frac{1}{2}$ is a mixed number.
negative numbers	numbers less than zero
numerator	the top number of a fraction, indicating the number of equal parts being considered *Example*: In the fraction $\frac{2}{3}$, the numerator is 2.
percent (%)	a special-case ratio in which the second term is always 100 *Example*: The ratio is written as a whole number followed by a percent sign, such as 25% which means the ratio of 25 to 100.
percent of change	the amount of change divided by the original amount *Example*: $\frac{\text{amount of change}}{\text{original amount}}$

percent of decrease	the percent the amount of decrease is of the original amount; also called the *discount* *Example*: $\frac{\text{amount of decrease}}{\text{original amount}}$
percent of increase	the percent the amount of increase is of the original amount *Example*: $\frac{\text{amount of increase}}{\text{original amount}}$
perimeter (*P*)	the length of the boundary around a figure; the distance around a polygon
place value	the position of a single digit in a whole number or decimal number containing one or more digits
positive numbers	numbers greater than zero
prime factorization	writing a number as the product of prime numbers *Example*: $24 = 2 \times 2 \times 2 \times 3 = 2^3 \times 3$
prime number	any whole number with only two factors, 1 and itself *Example*: 2, 3, 7, 11, etc.
principal	the amount of money on which interest is paid
product	the result of a multiplication *Example*: In $6 \times 8 = 48$, 48 is the product.

proportion	a mathematical sentence stating that two ratios are equal *Example*: The ratio of 1 to 4 equals 25 to 100, that is $\frac{1}{4} = \frac{25}{100}$.
quotient	the result of a division *Example*: In $42 \div 7 = 6$, 6 is the quotient.
rate/distance	calculations involving rates, distances, and time intervals, based on the distance, rate, time formula ($d = rt$); a ratio comparing two different units *Example*: miles per hour
ratio	the quotient of two numbers used to compare two quantities *Example*: The ratio of 3 to 4 is $\frac{3}{4}$.
reciprocals	two numbers whose product is 1; also called *multiplicative inverses* *Example*: Since $\frac{3}{4} \times \frac{4}{3} = 1$, the reciprocal of $\frac{3}{4}$ is $\frac{4}{3}$.
remainder	the whole number left after one number is divided by another number 

repeating decimal	a decimal in which one digit or a series of digits repeat endlessly *Example*: 0.3333333… or $0.\overline{3}$ 24.6666666… or $24.\overline{6}$ 5.27272727… or $5.\overline{27}$ 6.2835835… or $6.2\overline{835}$
rounded number	a number approximated to a specified place *Example*: A commonly used rule to round a number is as follows. • If the digit in the first place after the specified place is 5 or more, *round up* by adding 1 to the digit in the specified place ($\underline{4}61$ rounded to the nearest hundred is 500). • If the digit in the first place after the specified place is less than 5, *round down* by *not* changing the digit in the specified place ($\underline{4}41$ rounded to the nearest hundred is 400).
scientific notation	a shorthand method of writing very large or very small numbers using exponents in which a number is expressed as the product of a power of 10 and a number that is greater than or equal to one (1) and less than 10 *Example*: The number is written as a decimal number between 1 and 10 multiplied by a power of 10, such as $7.59 \times 10^5 = 759{,}000$. It is based on the idea that it is easier to read exponents than it is to count zeros. If a number is already a power of 10, it is simply written 10^{27} instead of 1×10^{27}.

simplest form	a fraction whose numerator and denominator have no common factor greater than 1 *Example*: The simplest form of $\frac{3}{6}$ is $\frac{1}{2}$.
simplify a fraction	write fraction in lowest terms or simplest form
solution	any value for a variable that makes an equation or inequality a true statement *Example*: In $y = 8 + 9$ $y = 17$ 17 is the solution.
solve	to find all numbers that make an equation or inequality true
substitute	to replace a variable with a numeral *Example*: $8(a) + 3$ $8(5) + 3$
sum	the result of an addition *Example*: In $6 + 8 = 14$, 14 is the sum.
terminating decimal	a decimal that contains a finite (limited) number of digits *Example*: $\frac{3}{8} = 0.375$ $\frac{2}{5} = 0.4$
unit	a precisely fixed quantity used to measure
unit price	the cost of one unit of a particular item, expressed in the unit in which the product is generally measured

unit rate	a rate with a denominator of 1; a rate for one unit of a given quantity *Example*: feet per second, miles per gallon, miles per hour, or cents per pound
value (of a variable)	any of the numbers represented by the variable
variable	any symbol that could represent a number
whole number	any number in the set {0, 1, 2, 3, 4, …}

Unit 2: Measurement

Introduction

We will discover that numbers can be written in several different ways. For instance, $\frac{1}{2}$ can be written several different ways and still have the same value.

$\frac{1}{2} = \frac{4}{8}$ $\frac{1}{2} = 0.5$ $\frac{1}{2} = \frac{5}{10}$ $\frac{1}{2} = 50\%$

It is important to know that the value of a number has not changed. Also, choosing which form is appropriate for a particular problem is important. With the presentation of numerous "real-world" situations, you will learn to choose among various methods of approaching and solving problems.

Lesson One Purpose

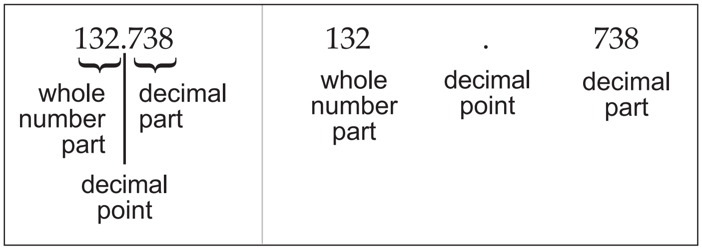
- Understand the relative size of integers, rational numbers, and real numbers. (MA.A.1.4.2)
- Understand that numbers can be represented in a variety of equivalent forms, including integers, fractions, decimals, percents, and exponents. (MA.A.1.4.4)
- Understand and use the real number system. (MA.A.2.4.2)
- Understand and explain the effects of addition, subtraction, multiplication, and division on real numbers, including exponents, and appropriate inverse relationships. (MA.A.3.4.1)
- Select and justify alternative strategies, such as using properties of numbers, including inverse, that allow operational shortcuts for computational procedures in real-world or mathematical problems. (MA.A.3.4.2)
- Add, subtract, multiply, and divide real numbers, including exponents, using appropriate methods of computing, such as mental mathematics, paper and pencil, and calculator. (MA.A.3.4.3)

Decimals and Fractions

Before considering the relationship between **decimals** and **fractions**, let's briefly review **place value**.

Place Value

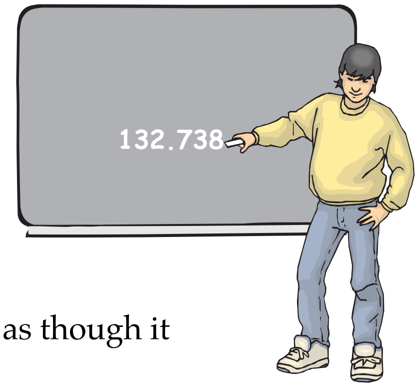
Place value is the position of a single **digit** in a whole number or decimal number. For example, the *decimal* 132.738 has three parts.



Use the chart below to see the place value of all the *digits* in the decimal 132.738.

1	3	2	.	7	3	8
hundreds	tens	ones	decimal point	tenths	hundredths	thousandths
100	10	1		$\frac{1}{10}$	$\frac{1}{100}$	$\frac{1}{1,000}$

Writing or Reading Decimals



To write or read a decimal in words:

- Write or read the **whole number** part in words.
- Write or read *and* for the **decimal point**.
- Write or read the decimal part in words as though it were a *whole number*.
- Write or read the place value of the last digit.

In words, 132.738 is written or said as *one hundred thirty-two* and *seven hundred thirty-eight* thousandths.

As a *fraction*, 132.738 is written as $132 \frac{738}{1,000}$.

Other examples:

Decimal	Words	Fraction
0.3	three tenths	$\frac{3}{10}$
0.51	fifty-one hundredths	$\frac{51}{100}$
0.07	seven hundredths	$\frac{7}{100}$
0.007	seven thousandths	$\frac{7}{1,000}$

Simplifying Fractions: Writing in Lowest Terms or Simplest Form

Frequently, we write decimals as fractions or as **mixed numbers** consisting of both a whole number and a fraction. Most times we are also asked to **simplify fractions** and show these answers in **simplest form**.

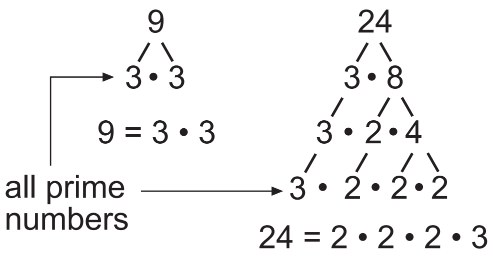
When is a fraction in *simplest form*?

Consider the fraction $\frac{9}{24}$.

Factor the **numerator**, 9, and the **denominator**, 24, of the fraction $\frac{9}{24}$.

Remember: A *factor* is a number that divides evenly into another number.

First find the **prime factorizations** of 9 and 24 to write each number as the product of **prime numbers**. A *prime number* is a whole number that has only two factors, 1 and itself.



numerator → denominator → $\frac{9}{24} = \frac{3 \times \cancel{3}}{2 \times 2 \times 2 \times \cancel{3}} = \frac{3}{8}$ We cancel one 3 in the numerator and one 3 in the denominator and get $\frac{3}{8}$.

Another way to simplify $\frac{9}{24}$ and write in simplest form is to divide the numerator and denominator by their **greatest common factor (GCF)**.

Remember: The *greatest common factor* or *GCF* is the largest of the **common factors** of two or more numbers. *Common factors* are factors of two or more numbers.

$$\frac{9 \div 3}{24 \div 3} = \frac{3}{8}$$

> **To summarize:** A fraction is *reduced* or in *simplest form* when the greatest number that divides both the numerator and denominator is 1. In other words, the fraction's numerator and denominator have no common factor greater than 1.

Another example:

Reduce $\frac{28}{40}$.

- Factor method: $\frac{28}{40} = \frac{\cancel{2} \times \cancel{2} \times 7}{\cancel{2} \times \cancel{2} \times 2 \times 5} = \frac{7}{10}$
- Greatest common factor (GCF) method: $\frac{28 \div 4}{40 \div 4} = \frac{7}{10}$

Converting Decimals to Fractions

Now that we have practiced writing fractions in simplest form, let's start with decimals and convert the decimals to fractions written in simplest form. Either the *factor method* or the *greatest common factor method* can be used to write the fraction in simplest form. Consider the following examples:

$$0.125 = \frac{125}{1{,}000} = \frac{1}{8}$$

$$22.5 = 22\frac{5}{10} = 22\frac{1}{2}$$

$$0.25 = \frac{25}{100} = \frac{1}{4}$$

Converting Fractions to Decimals

To change fractions to decimals we use long division. See below. In the fraction $\frac{3}{8}$, the bar separating the 3 and the 8 is called the *fraction bar*, so $\frac{3}{8}$ is read as 3 *divided by* 8. In $\frac{3}{2}$, it is read as 3 *divided by* 2.

Examples:

$$\frac{3}{8} = ? \quad \begin{array}{r} 0.375 \\ 8\,\overline{)3.000} \\ \underline{2\,4} \\ 60 \\ \underline{56} \\ 40 \\ \underline{40} \\ 0 \end{array} \qquad\qquad \frac{3}{2} = ? \quad \begin{array}{r} 1.5 \\ 2\,\overline{)3.0} \\ \underline{2} \\ 10 \\ \underline{10} \\ 0 \end{array}$$

See the example below. The division does not come out evenly—the remainder is not 0. The **quotient** is a **repeating decimal**. A *repeating decimal* is a decimal in which one digit or a series of digits repeat endlessly. In this example, the answer was **rounded** to the nearest hundredth.

In the example below, $1\frac{1}{4}$ is a *mixed number*. The mixed number was first changed to its **equivalent improper fraction** before converting the fraction into a decimal using long division.

$$\frac{1}{6} = ? \quad \begin{array}{r} 0.166 = 0.17 \\ 6\,\overline{)1.000} \\ \underline{6} \\ 40 \\ \underline{36} \\ 40 \\ \underline{36} \\ 4 \end{array} \qquad\qquad 1\,\frac{1}{4} = \frac{5}{4} = ? \quad \begin{array}{r} 1.25 \\ 4\,\overline{)5.00} \\ \underline{4} \\ 10 \\ \underline{8} \\ 20 \\ \underline{20} \\ 0 \end{array}$$

Practice

Change each **decimal** *to a* **fraction** *or* **mixed number**. *Write each answer in* **simplest form**.

1. 0.25 ______________

2. 0.02 ______________

3. 0.2 ______________

4. 0.625 ______________

5. 2.6 ______________

6. 0.5 ______________

7. 0.05 ______________

8. 0.18 ______________

9. 3.6 ______________

10. 1.25 ______________

11. 0.008 ______________

Change each **fraction** *to a* **decimal**. *If necessary,* **round repeating decimals to the nearest hundredth**.

12. $\frac{3}{5}$ _______________

13. $\frac{1}{8}$ _______________

14. $\frac{3}{4}$ _______________

15. $\frac{1}{2}$ _______________

16. $\frac{4}{9}$ _______________

17. $\frac{5}{4}$ _______________

18. $\frac{7}{6}$ _______________

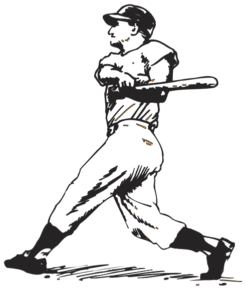
19. $\frac{11}{20}$ _______________

20. $1\frac{5}{8}$ _______________

21. $\frac{2}{3}$ _______________

Complete the following. In figuring **batting averages** *for softball and baseball players, we divide the player's number of "***hits***" by the number of times the player is "***at bat***." Batting averages are expressed as a* **decimal rounded to the nearest thousandth**.

Use the **data** *for these top five softball players to* **figure their batting averages**.

	Top Five			**Batting Average** (in thousandths)
	Name	**Number of Hits**	**Number of Times at Bat**	
22.	Adkins	6	16	________ 
23.	Bruce	6	18	________
24.	George	6	20	________
25.	Miller	5	14	________
26.	Thomas	8	20	________

27. Who had the best batting average? ________

Practice

Use the list below to write the correct term for each definition on the line provided.

decimal number
decimal point
digit
factor
fraction
mixed number
place value
prime factorization
prime number
simplest form
whole number

______________________ 1. any number representing some part of a whole; of the form $\frac{a}{b}$

______________________ 2. a fraction whose numerator and denominator have no common factor greater than 1

______________________ 3. the dot dividing a decimal number's whole part from its fractional part

______________________ 4. any number in the set {0, 1, 2, 3, 4, ...}

______________________ 5. any number written with a decimal point in the number

______________________ 6. any whole number with only two factors, 1 and itself

______________________ 7. a number that consists of both a whole number and a fraction

______________________ 8. a number or expression that divides exactly another number

______________________ 9. any one of the 10 symbols 0, 1, 2, 3, 4, 5, 6, 7, 8, or 9

______________________ 10. writing a number as the product of prime numbers

______________________ 11. the position of a single digit in a whole number or decimal number containing one or more digits

Practice

Match each definition with the correct term. Write the letter on the line provided.

______ 1. the result of a division

______ 2. the same number expressed in different forms

______ 3. the bottom number of a fraction, indicating the number of equal parts a whole was divided into

______ 4. write fraction in lowest terms or simplest forms

______ 5. a number approximated to a specified place

______ 6. the top number of a fraction, indicating the number of equal parts being considered

______ 7. a number that is a factor of two or more numbers

______ 8. the largest of the common factors of two or more numbers

______ 9. a fraction that has a numerator greater than or equal to the denominator

______ 10. a decimal in which one digit or a series of digits repeat endlessly

A. common factor

B. denominator

C. equivalent (form of a number)

D. greatest common factor (GCF)

E. improper fraction

F. numerator

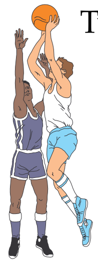
G. quotient

H. repeating decimal

I. rounded number

J. simplify a fraction

Comparing Fractions and Decimals



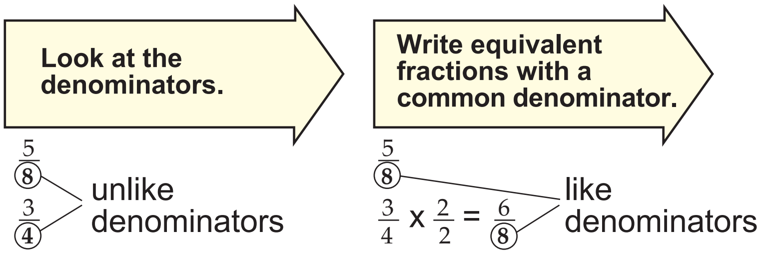
Two boys played basketball over the weekend. George bragged that he sank 5 baskets out of 8. Jamaar boasted that he sank 3 baskets out of 4. Which boy really has "bragging rights"?

This problem can be solved by finding which fraction is larger. Is it $\frac{5}{8}$ or $\frac{3}{4}$? There are a few ways to tell, so let's examine three of them.

Method 1: The Common Denominator Method

The fractions $\frac{5}{8}$ and $\frac{3}{4}$ obviously have different denominators. They need *like* denominators to be able to be easily compared. In this case, 4 is a factor of 8, therefore 8 is a **common denominator** of both fractions. We can change the *appearance* of $\frac{3}{4}$ and keep its value.

Comparing Fractions



$\frac{3}{4} \times \frac{2}{2} = \frac{6}{8}$

Think. What number times 4 equals 8? Answer: 2. To keep the *equivalent* value of the fraction $\frac{3}{4}$ and to change its denominator to 8, multiply it by $\frac{2}{2}$. The fraction $\frac{2}{2}$ is another name for the number 1. If you multiply a number by 1, nothing happens to it—the result is the original number. The number 1 is the *identity element* for multiplication, also called the *property of multiplying by 1*. See properties in Unit 1.

We say that $\frac{6}{8}$ and $\frac{3}{4}$ are equivalent fractions. The fractions have the same value.

Now we have $\frac{5}{8}$ and $\frac{6}{8}$ which can be easily compared.

Compare the numerators.

The fractions compare the same way the numerators compare.

$\frac{\textcircled{5}}{8}$

$\frac{\textcircled{6}}{8}$

$5 < 6$

Since $5 < 6$,
then $\frac{5}{8} < \frac{6}{8}$,
therefore $\frac{5}{8} < \frac{3}{4}$.

For fractions with *like* denominators, the *larger fraction* is the one with the *larger numerator*.

Therefore $\frac{5}{8} < \frac{6}{8}$ or $\frac{5}{8} < \frac{3}{4}$ or $\frac{3}{4} > \frac{5}{8}$.

Here is another example using the *common denominator* method:

Which is larger? $\frac{8}{9}$ or $\frac{7}{8}$? The denominators, 9 and 8, are both factors of 72. Let's rewrite both fractions so that they both have a *like* denominator of 72.

$\frac{8}{9} \bullet \frac{8}{8} = \frac{64}{72}$ *and* $\frac{7}{8} \bullet \frac{9}{9} = \frac{63}{72}$

$\frac{8}{8}$ is another name for 1.

$\frac{64}{72} > \frac{63}{72}$

$\frac{9}{9}$ is another name for 1.

therefore

$\frac{8}{9} > \frac{7}{8}$

Tip: You may wish to draw a "big one" around fractions equivalent to one as a reminder of their value.

$$\frac{8}{9} \bullet \boxed{\frac{8}{8}} = \frac{64}{72} = \frac{8}{9}$$

or

$$\frac{7}{8} \bullet \boxed{\frac{9}{9}} = \frac{63}{72} = \frac{7}{8}$$

Method 2: The Change Fractions into Decimals Method

Let's go back to the original problem about George and Jamaar. Another way to compare fractions is to express them as decimals, and then compare the decimals.

Remember: To change a fraction to a decimal, we divide the denominator into the numerator. For example, the fraction $\frac{3}{4}$ is read as *3 divided by 4*, and the fraction $\frac{5}{8}$ is read as *5 divided by 8*.

$$\begin{array}{r} 0.75 \\ 4\overline{)3.0} \\ \underline{2\ 8} \\ 20 \end{array} \qquad\qquad \begin{array}{r} 0.625 \\ 8\overline{)5.000} \\ \underline{4\ 8} \\ 20 \\ \underline{16} \\ 40 \end{array}$$

$$\frac{3}{4} = 0.75 \textit{ and } \frac{5}{8} = 0.625$$

To compare decimals, use place value to compare the digits in the same place. The decimal 0.75 is the same as the decimal 0.750. Just because a decimal is "longer" than another decimal, do *not* assume that it is larger!

$$0.75 = 0.750 \textit{ and}$$
$$0.750 > 0.625 \textit{ because } 7 > 6 \textit{ so } \frac{3}{4} > \frac{5}{8}$$

Comparing Decimal Numbers

Line up the numbers using their decimal points. Start at the left and compare digits in the same place value.

compare place value

$$\begin{array}{l} 0.750 \\ 0.625 \end{array}$$

Find the digits in the first place where they are different, and then compare.

different digit in same place value

$$\begin{array}{ll} 0.\textcircled{7}50 & \\ 0.\textcircled{6}25 & \quad 7 > 6 \end{array}$$

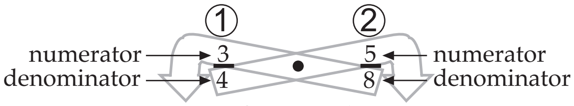
The decimal numbers compare the same way the digits compare.

Since 7 > 6,
then 0.750 > 0.625.

0.750 > 0.625
or
0.625 < 0.750

Method 3: The Cross Products Method

We can compare fractions by using **cross products**. *Cross products* are the **product** of one numerator and the *opposite* denominator in a pair of fractions. If the cross products are equal, the fractions are equivalent.



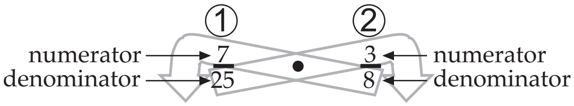
8 x 3   4 x 5

24   20

Since 24 $>$ 20,

then $\frac{3}{4} > \frac{5}{8}$.

Here is another example using cross products.



8 x 7   25 x 3

56   75

Since 56 $<$ 75,

then $\frac{7}{25} < \frac{3}{8}$.

Practice

Fill in the blank with >, <, *or* =.

1. $\frac{1}{5}$ ________ $\frac{3}{10}$

2. $\frac{1}{3}$ ________ $\frac{2}{7}$

3. $\frac{5}{4}$ ________ $\frac{8}{7}$

4. $\frac{15}{12}$ ________ $\frac{5}{4}$

5. 0.2 ________ 0.18

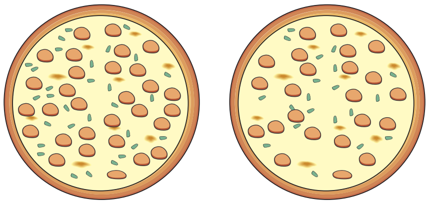
6. $\frac{1}{8}$ ________ 0.29

7. 6.58 ________ $6\,\frac{58}{100}$

8. $\frac{1}{2}$ ________ 0.59

9. $\frac{69}{1{,}000}$ ________ $\frac{6}{100}$

10. Your mom orders two large pizzas. She cuts one pizza into 6 parts, and your brother eats 4 pieces. She cuts the other pizza into 8 parts, and you eat 5 pieces.



Who ate the most pizza and why? ______________________________

____

____

____

Order these **fractions** *from* **least** *to* **greatest**.

11. $\frac{1}{3}, \frac{1}{2}, \frac{1}{4}$ ____________________

12. $\frac{1}{3}, \frac{2}{9}, \frac{2}{5}$ ____________________

13. $\frac{3}{7}, \frac{4}{9}, \frac{12}{21}$ ____________________

Fill in the blanks with <, >, or =. Use the given **value** *of each* **variable** *below to* **evaluate each expression**.

$a = -4$ $b = -5$ $c = 3$ $d = 9$

14. $\frac{c-1}{d+1}$ _______ $\frac{1}{3}$

15. $\frac{a+d}{c}$ _______ $\frac{a+b}{b}$

Remember: Rules for dividing **integers**

same signs → **positive quotient**

$\frac{6}{2} = 3$

$\frac{-6}{-2} = 3$

different signs → **negative quotient**

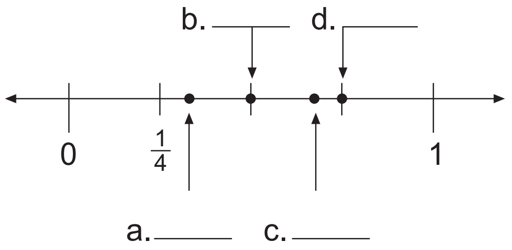
$\frac{-6}{2} = -3$

$\frac{6}{-2} = -3$

Complete the following.

16. Write each fraction on the correct point on the number line.

$\frac{1}{2}, \frac{3}{4}, \frac{2}{3}, \frac{1}{3}$



17. Explain in your own words why $-\frac{7}{8}$ is smaller than $-\frac{1}{2}$. ________

____

____

____

18. In your own words, describe two different ways to compare $\frac{7}{8}$ and 0.8.

____

____

____

Renaming Fractions and Mixed Numbers

Simplifying a Fraction

Previously, we reviewed the process of *reducing* or simplifying a fraction and writing it in *simplest form*. We reduced a fraction to its *lowest terms* so that its numerator and denominator have no common factor greater than 1 and then wrote it in *simplest form*. This extends to simplifying fractions in algebraic *expressions* as well. Reducing fractions is really just looking for ones!

Study these examples:

Factor method **GCF method**

Example: $\frac{15}{25} = \frac{\cancel{5} \bullet 3}{\cancel{5} \bullet 5} = \frac{3}{5}$ or $\frac{15}{25} = \frac{15 \div 5}{25 \div 5} = \frac{3}{5}$

Example: $\frac{12x^3y^2}{16x^4y} = \frac{\cancel{4} \bullet 3 \bullet \cancel{x} \bullet \cancel{x} \bullet \cancel{x} \bullet y \bullet \cancel{y}}{\cancel{4} \bullet 4 \bullet \cancel{x} \bullet \cancel{x} \bullet \cancel{x} \bullet x \bullet \cancel{y}} = \frac{3y}{4x}$

Example: $\frac{-34mn}{68m^2} = \frac{-\cancel{34} \bullet \cancel{m} \bullet n}{\cancel{34} \bullet 2 \bullet \cancel{m} \bullet m} = \frac{-1n}{2m} = \frac{-n}{2m}$

Remember: $-1n = -n$

We *always* reduce fractions and write them in simplest form if we can!

Renaming a Fraction

In the last section we found that we could **change the appearance of fractions by multiplying by one**. The number 1 is the identity element for multiplication. Multiplying a number by 1 does not change a number's value.

Study the following examples:

$$\frac{5}{7} \bullet \boxed{\frac{2}{2}} = \frac{10}{14} \qquad \frac{5}{7} \bullet \boxed{\frac{3}{3}} = \frac{15}{21} \qquad \frac{5}{7} \bullet \boxed{\frac{4}{4}} = \frac{20}{28} \qquad \frac{5}{7} \bullet \boxed{\frac{x}{x}} = \frac{5x}{7x}$$

We know that $\frac{2}{2}$, $\frac{3}{3}$, $\frac{4}{4}$, and $\frac{x}{x}$ are all equal to 1. Multiplying $\frac{5}{7}$ times any of those fractions equal to 1 will not change the value of $\frac{5}{7}$. Therefore, we know that $\frac{5}{7}$, $\frac{10}{14}$, $\frac{15}{21}$, $\frac{20}{28}$, and $\frac{5x}{7x}$ are all equivalent fractions.

Here is another example:

> Change $\frac{2x}{9y}$ to an equivalent fraction that has 18y in the denominator.
>
> 9 x 2 = 18, so the "1" we should use should be $\frac{2}{2}$
>
> $$\frac{2x}{9y} \bullet \boxed{\frac{2}{2}} = \frac{4x}{18y}$$
>
> $\frac{2x}{9y}$ is equivalent to $\frac{4x}{18y}$

Changing Improper Fractions to Mixed Numbers

We say that these fractions are *improper fractions*: $\frac{7}{5}$, $\frac{11}{6}$, $\frac{15}{8}$, $\frac{50}{7}$

Improper fractions are fractions where the numerator is larger than the denominator.

There is nothing wrong with this, but sometimes rewriting these numbers can help us get a *feel* for their size.

Study the following examples:

Example: $\frac{18}{5}$

$$\begin{array}{r} 3 \\ 5\,\overline{)\,18} \\ \underline{15} \\ 3 \end{array} \leftarrow$$

number of fifths left over

therefore, $\frac{18}{5} = 3\frac{3}{5}$

write **remainder** of 3 over **divisor** of 5

Example: $\frac{27}{10}$

$$\begin{array}{r} 2 \\ 10\,\overline{)\,27} \\ \underline{20} \\ 7 \end{array} \leftarrow$$

number of tenths left over

therefore, $\frac{27}{10} = 2\frac{7}{10}$

write *remainder* of 7 over *divisor* of 10

Example: $\frac{18}{3}$

$$\begin{array}{r} 6 \\ 3\,\overline{)\,18} \\ \underline{18} \\ 0 \end{array} \leftarrow$$

no number is left over, so there is no remainder

therefore, $\frac{18}{3} = 6$

Example: $\frac{38}{8}$

$$\begin{array}{r} 4 \\ 8\,\overline{)\,38} \\ \underline{32} \\ 6 \\ \uparrow \end{array}$$

number of eighths left over

therefore, $\frac{38}{8} = 4\frac{6}{8}$

but $\frac{6}{8} = \frac{\cancel{2} \cdot 3}{\cancel{2} \cdot 4} = \frac{3}{4}$

$\frac{38}{8} = 4\frac{3}{4}$

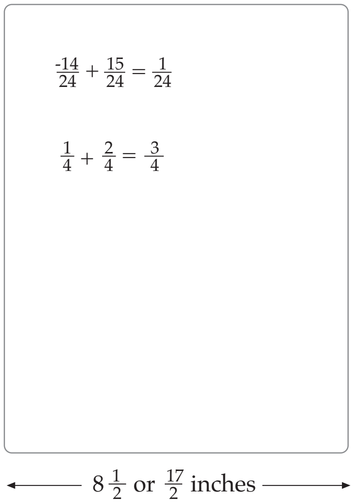
write remainder of 6 over divisor of 8 in simplest form $\frac{6}{8} = \frac{3}{4}$

We must reduce!

Reversing the Process: Writing Mixed Numbers and Whole Numbers as Fractions

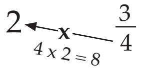
Mixed numbers like $8\frac{1}{2}$ are useful. They help us understand the size of a number. Most of us can understand that most letter-sized printer paper is $8\frac{1}{2}$ inches wide. However, we would have trouble visualizing how wide $\frac{17}{2}$ inches would be. Yet both numbers represent the same amount.

In the next few sections we will discover that while mixed numbers have their place in our world, they are difficult to deal with when we have to add, subtract, multiply, or divide them. Here the improper fraction is useful.

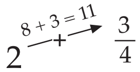


Study the example below to review how to change *mixed numbers* to *improper fractions*:

$2\frac{3}{4}$



Multiply 2 and 4. (This gives the number of fourths in 2.)

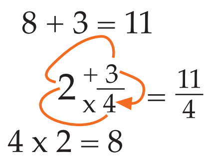


Now add the answer 8 and 3. (This gives the number of fourths in $2\frac{3}{4}$.)

Therefore,

$$2\frac{3}{4} = \frac{11}{4}.$$

Or visualize it like this:



Start with 4 times the 2, plus the 3, equals 11, and put the 11 over the first number.

So, $2\frac{3}{4} = \frac{11}{4}$.

Study the example below to review how to change *whole numbers* to *fractions*.

Suppose that we want to change the whole number 5 to fourths.

Remember that $5 = \frac{5}{1}$

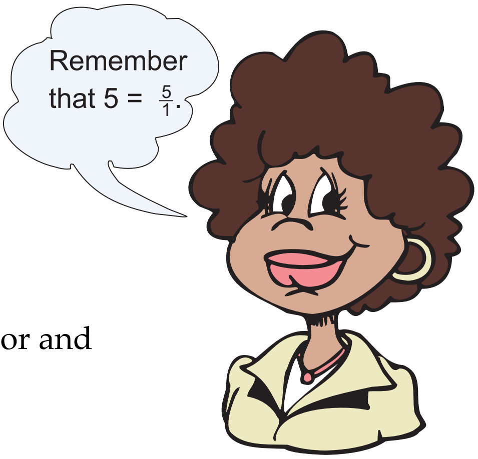
Since we do *not* change the value of the fraction, if we multiply the fraction by 1, we can multiply $\frac{5}{1}$ times $\frac{4}{4}$.

$$\frac{5}{1} \bullet \frac{4}{4} = \frac{20}{4}$$

Notice we multiply both numerator and denominator by 4.

Therefore,

$$5 = \frac{20}{4}.$$



Practice

Change these **whole numbers** *to* **eighths***.*

1. 2 ____________
2. 5 ____________
3. 7 ____________

Change these **mixed numbers** *to* **fractions***.*

4. $2\frac{1}{3}$ ____________
5. $6\frac{5}{6}$ ____________
6. $5\frac{3}{4}$ ____________

Reduce *each fraction and* **write in simplest form***.*

7. $\frac{9}{42}$ ____________
8. $\frac{36}{54}$ ____________
9. $\frac{8}{28}$ ____________
10. $\frac{2xy}{8y^2}$ ____________
11. $\frac{3x^2y}{18y}$ ____________
12. $\frac{28p^2q^2}{42p^3q^3}$ ____________
13. $\frac{18mn}{9mn}$ ____________

Write **three** *fractions that are* **equivalent** *to the* **given fraction** *or* **whole number***.*

14. $\frac{1}{4}$ ____________________
15. $\frac{3}{5}$ ____________________
16. $\frac{10}{24}$ ____________________
17. 10 ____________________

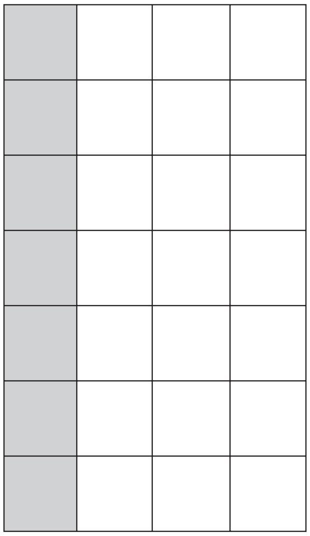
Circle the letter of the correct answer.

18. Which of the following is equivalent to $\frac{7}{15}$? ____________

    a. $\frac{24}{60}$ b. $\frac{3}{5}$ c. $\frac{14}{17}$ d. $\frac{21}{45}$

19. **Estimate** the fraction of the rectangle that is *not* shaded.

    a. $\frac{5}{6}$ b. $\frac{4}{7}$ c. $\frac{3}{4}$ d. $\frac{9}{10}$



Answer the following.

20. I am a fraction less than one. I am reduced. My numerator and denominator are both *prime numbers* between 25 and 36. What am I?

Remember: A prime number is any whole number with only two factors, 1 and itself.

Answer: ____________

Practice

Use the list below to write the correct term for each definition on the line provided.

common denominator
cross product
divisor
estimation
expression
integers
remainder
value (of a variable)
variable

____________________ 1. the whole number left after one number is divided by another number

____________________ 2. any symbol that could represent a number

____________________ 3. the product of one numerator and the opposite denominator in a pair of fractions

____________________ 4. a collection of numbers, symbols, and/or operation signs that stands for a number

____________________ 5. any of the numbers represented by the variable

____________________ 6. a number by which another number, the dividend, is divided

____________________ 7. the numbers in the set {… , -4, -3, -2, -1, 0, 1, 2, 3, 4, …}

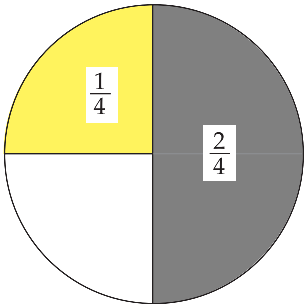
____________________ 8. a common multiple of two or more denominators

____________________ 9. the use of rounding and/or other strategies to determine a reasonably accurate approximation without calculating an exact answer

Adding and Subtracting Fractions

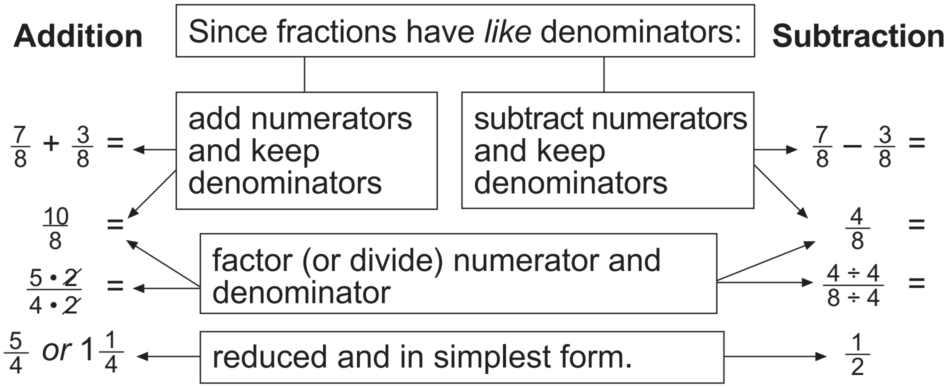
Adding and Subtracting Fractions with Like Denominators

We know that one quarter (one fourth) plus two quarters (two fourths) adds to three quarters (three fourths). Mathematically, this sentence translates to $\frac{1}{4} + \frac{2}{4} = \frac{3}{4}$.



To add (or subtract) fractions with *like* denominators:

- add (or subtract) the numerators
- write the **sum** (or **difference**) over the denominator
- write final answer in simplest form.



Remember: $\frac{5}{4}$ is an *improper fraction* which has been simplified and reduced and is in simplest form. However, it may also be renamed by writing it as $1\frac{1}{4}$, which is called a *mixed number* in simplest form.

The rule for adding (or subtracting) fractions with *like* denominators can be thought of in this way:

> Add (or subtract) the top.
> Keep the bottom.
> *Always* reduce to simplest form.

Study the next two examples and see if you can understand how the rule is used.

How to Add Fractions with Like Denominators

Example:

$\frac{5}{8} + \frac{1}{8} =$	←	denominators are the same, add numerators		
$\frac{6}{8} =$	←			
$\frac{3 \cdot \cancel{2}}{4 \cdot \cancel{2}} =$	←	factor (or divide) numerator and denominator	→	$\frac{6 \div 2}{8 \div 2} =$
$\frac{3}{4}$	←	reduced and in simplest form	→	$\frac{3}{4}$

Example:

$\frac{7}{14} + \frac{11}{14} =$	←	denominators are the same, add numerators		
$\frac{18}{14} =$	←			
$\frac{9 \cdot \cancel{2}}{7 \cdot \cancel{2}} =$	←	factor (or divide) numerator and denominator	→	$\frac{18 \div 2}{14 \div 2} =$
$\frac{9}{7}$ *or* $1\frac{2}{7}$	←	reduced and in simplest form	→	$\frac{9}{7}$ *or* $1\frac{2}{7}$

Here are several more examples that use the rule.

How to Subtract Fractions with Like Denominators

Example:

$\frac{10}{7} - \frac{3}{7} =$ ← subtract numerators

$\frac{7}{7}$ *or* 1 ← reduced and in simplest form

How to Add a Negative and a Positive Fraction

Example:

$\frac{-14}{24} + \frac{15}{24} =$ ← add numerators

$\frac{1}{24}$ ← reduced and in simplest form

How to Add Mixed Numbers with Like Denominators

Example:

$1\frac{5}{7} + 3\frac{3}{7} =$

rewrite as an improper fraction

$\frac{12}{7} + \frac{24}{7} =$

add numerators

$\frac{36}{7}$ *or* $5\frac{1}{7}$

reduced and in simplest form

How to Subtract Mixed Numbers with Like Denominators

Example:

$6\frac{1}{4} - 2\frac{3}{4} =$

rewrite as an improper fraction

$\frac{25}{4} - \frac{11}{4} =$

subtract numerators

$\frac{14}{4} =$

factor (or divide) numerator and denominator → $\frac{14 \div 2}{4 \div 2} =$

$\frac{\cancel{2} \cdot 7}{\cancel{2} \cdot 2} =$

$\frac{7}{2}$ *or* $3\frac{1}{2}$ ← reduced and in simplest form → $\frac{7}{2}$ *or* $3\frac{1}{2}$

Fractions and Equations

Sometimes we have to deal with fractions when we **solve equations**. *Equations* are mathematical sentences that equate one expression to another. Study this example and see if you remember all the steps from Unit 1.

How to Handle Fractions in Solving Equations

Example:

$$
\begin{aligned}
x + \tfrac{1}{5} &= \tfrac{7}{5}\\
x + \tfrac{1}{5} - \tfrac{1}{5} &= \tfrac{7}{5} - \tfrac{1}{5} && \leftarrow \text{subtract } \tfrac{1}{5} \text{ from both sides}\\
x &= \tfrac{6}{5} \textit{ or } 1\tfrac{1}{5} && \leftarrow \text{reduced and in simplest form}
\end{aligned}
$$

Does the **solution** of $\frac{6}{5}$ or $1\frac{1}{5}$ make the equation true? To check, **substitute** $\frac{6}{5}$ for the variable x in the original equation.

$$
\begin{aligned}
x + \tfrac{1}{5} &= \tfrac{7}{5}\\
\tfrac{6}{5} + \tfrac{1}{5} &= \tfrac{7}{5}\\
\tfrac{7}{5} &= \tfrac{7}{5}\\
&\textit{or} && \leftarrow \textit{It checks!}\\
1\tfrac{1}{5} &= 1\tfrac{1}{5}
\end{aligned}
$$

Writing Negative Fractions

Many times in algebra we have to deal with **negative numbers** and *negative* fractions. You need to be aware of the following:

Negative fractions can be written in three ways.

$$\frac{5}{-7},\ \frac{-5}{7},\ \text{and}\ -\frac{5}{7}$$

All of these fractions are equivalent.

Here is an example which contains negative fractions:

How to Add Negative Fractions with Like Denominators

Example:

$\frac{5}{-7} + \frac{-3}{7} =$ ← since denominators are *not* the same,

$\frac{-5}{7} + \frac{-3}{7} =$ ← rewrite the fractions so the signs are in the numerators

$\frac{-8}{7}$ *or* $-1\frac{1}{7}$ ← add numerators

← reduced and in simplest form

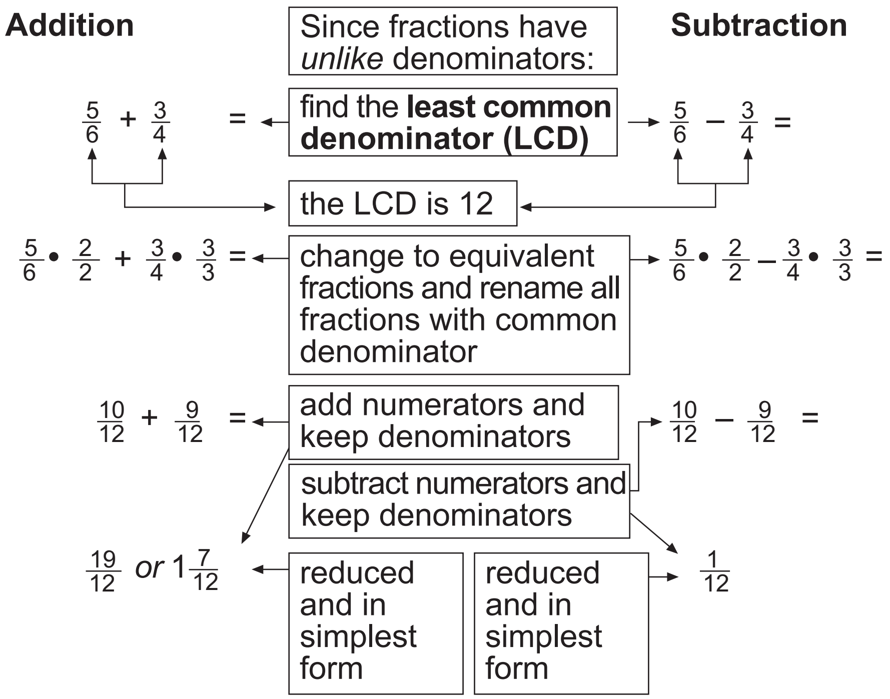
Note: When a negative improper fraction is written as a mixed number, the entire mixed number takes the negative sign.

$$\frac{-8}{7} = -1\frac{1}{7}$$

Adding and Subtracting Fractions with Unlike Denominators

To add (or subtract) fractions with *unlike* denominators:

- rename all fractions so that there is a common denominator
- add (or subtract) the numerators
- write the *sum* (or *difference*) over the denominator
- always reduce to simplest form.



Study the following examples.

How to Add Fractions with Unlike Denominators

Example:

$$\frac{3}{4} + \frac{2}{5} =$$

find LCD

$$\frac{3}{4} \bullet \frac{5}{5} + \frac{2}{5} \bullet \frac{4}{4} =$$

rewrite with common denominators

$$\frac{15}{20} + \frac{8}{20} =$$

add numerators

$$\frac{23}{20} \textit{ or } 1\frac{3}{20}$$

reduced and in simplest form

How to Subtract Mixed Numbers with Unlike Denominators

Example:

$$8\frac{1}{2} - 3\,\frac{3}{4} =$$

rewrite as an
improper fraction

$$\frac{17}{2} - \frac{15}{4} =$$

find LCD

$$\frac{17}{2} \bullet \frac{2}{2} - \frac{15}{4} =$$

rewrite with common
denominators

$$\frac{34}{4} - \frac{15}{4} =$$

subtract numerators
or
rewrite to add using
additive inverse of
second fraction

$$\frac{34}{4} + \frac{\mathbf{-15}}{\mathbf{4}} =$$

$$\frac{19}{4} \textit{ or } 4\frac{3}{4}$$

reduced and in
simplest form

$$\frac{19}{4} \textit{ or } 4\frac{3}{4}$$

How to Combine Negative Mixed Numbers

Example:

$-3\frac{3}{7} - 1\frac{1}{3} \quad =$

rewrite as improper fractions

$\frac{-24}{7} - \frac{4}{3} \quad =$

find LCD

$\frac{-24}{7} \bullet \frac{3}{3} - \frac{4}{3} \bullet \frac{7}{7} =$

rewrite with common denominators

$\frac{-72}{21} - \frac{28}{21} \quad =$

subtract numerators

or

rewrite to add using *additive inverse* of second fraction

$\frac{-72}{21} + \mathbf{\frac{-28}{21}} =$

$\frac{-100}{21}$ *or* $-4\frac{16}{21}$

reduced and in simplest form

$\frac{-100}{21}$ *or* $-4\frac{16}{21}$

How to Subtract Whole Numbers and Fractions

Example:

$4 - \frac{5}{6} \quad =$

rewrite the whole number as improper fraction

$\frac{4}{1} - \frac{5}{6} \quad =$

find LCD

$\frac{4}{1} \bullet \frac{6}{6} - \frac{5}{6} \quad =$

rewrite with common denominators

$\frac{24}{6} - \frac{5}{6} \quad =$

subtract numerators

or

rewrite to add using additive inverse of second fraction

$\frac{24}{6} + \mathbf{\frac{-5}{6}} \quad =$

$\frac{19}{6}$ *or* $3\frac{1}{6}$

reduced and in simplest form

$\frac{19}{6}$ *or* $3\frac{1}{6}$

On the following page, the last example requires us to use many skills. See if you can follow all the steps in solving the equation. You will have to remember skills from Unit 1.

How to Solve Equations with Mixed Numbers Using Additive Inverse

Remember: To solve an equation, you must get the *variable*—or any symbol that can represent a number—alone on one side of the equals sign. To do this you need to *undo* any operations on the variable.

Example:

$$x + 5\tfrac{1}{6} = 3\tfrac{2}{3}$$

undo the addition of $5\tfrac{1}{6}$

subtract $5\tfrac{1}{6}$ from both sides

$$x + 5\tfrac{1}{6} - 5\tfrac{1}{6} = 3\tfrac{2}{3} - 5\tfrac{1}{6}$$

rewrite mixed numbers as improper fractions and find LCD

$$x = \frac{11}{3} - \frac{31}{6}$$

rewrite with common denominators

$$x = \frac{11}{3} \bullet \frac{2}{2} - \frac{31}{6}$$

$$x = \frac{22}{6} - \frac{31}{6}$$

subtract numerators

or

rewrite to add using additive inverse of second fraction

$$x = \frac{22}{6} + \mathbf{\frac{-31}{6}}$$

$$x = \frac{-9}{6}$$

factor (or divide) numerator and denominator to write in lowest terms

$$x = \frac{\cancel{3} \bullet -3}{\cancel{3} \bullet 2}$$

$$x = \frac{-9 \div 3}{6 \div 3}$$

$$x = \frac{-3}{2}$$

reduced and in simplest form

$$x = -1\tfrac{1}{2}$$

To check solution:

$$x + 5\tfrac{1}{6} = 3\tfrac{2}{3}$$

substitute the solution for the variable

$$\mathbf{-1\tfrac{1}{2}} + 5\tfrac{1}{6} = 3\tfrac{2}{3}$$

rewrite mixed numbers as improper fractions and find LCD

$$-\frac{3}{2} + \frac{31}{6} = 3\tfrac{2}{3}$$

rewrite with common denominators

$$-\frac{3}{2} \bullet \frac{3}{3} + \frac{31}{6} = 3\tfrac{2}{3}$$

add numerators

$$-\frac{9}{6} + \frac{31}{6} = 3\tfrac{2}{3}$$

$$\frac{22}{6} = 3\tfrac{2}{3}$$

reduced and in simplest form

$$3\tfrac{4}{6} = 3\tfrac{2}{3}$$

$$3\tfrac{2}{3} = 3\tfrac{2}{3}$$

It checks!

Practice

Answer the following. Write in **simplest form**.

1. $\frac{5}{6} + \frac{7}{6}$ = ______________
2. $\frac{5}{8} + \frac{5}{8}$ = ______________
3. $\frac{7}{9} + \frac{8}{9}$ = ______________
4. $\frac{24}{25} - \frac{19}{25}$ = ______________
5. $\frac{4}{9} + \frac{1}{6}$ = ______________
6. $\frac{4}{5} + \frac{3}{8}$ = ______________
7. $5\frac{1}{4} + 1\frac{3}{4}$ = ______________
8. $8\frac{3}{4} - 3\frac{1}{2}$ = ______________
9. $\frac{-9}{5} - \frac{11}{5}$ = ______________
10. $8\frac{1}{5} - 2\frac{1}{4}$ = ______________

Solve these equations. *Write each answer in* **simplest form**. *Show all your work.*

11. $x + \frac{2}{3} = \frac{7}{2}$

12. $x + 3\frac{1}{4} = 7$

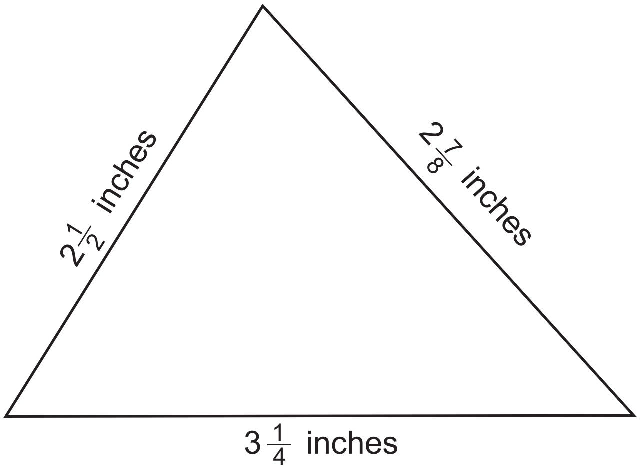
13. $x + \frac{3}{8} = 2\frac{1}{2}$

14. $x - \frac{7}{8} = 2\frac{1}{4}$

15. Find the **perimeter (*P*)** of this triangle.

Remember: *Perimeter* means the distance around a figure or the *sum* of the length of the boundary.

Answer: ____________ inches



Multiplying and Dividing Fractions

Multiplying Fractions

To multiply two fractions—(if need be, rewrite mixed numbers as improper fractions)—multiply the numerators and multiply the denominators. Always reduce!

Study the following three examples to review multiplying fractions:

How to Multiply Fractions

Example:

$\frac{4}{9} \bullet \frac{2}{3}$ = ← multiply numerators
multiply denominators

$\frac{8}{27}$ ← reduced and in simplest form

Example:

$\frac{5}{6} \bullet \frac{2}{5}$ = ← multiply numerators
multiply denominators

$\frac{10}{30}$ = ← factor (or divide) numerator
and denominator to

$\frac{\cancel{10} \text{ x } 1}{\cancel{10} \text{ x } 3}$ = write in lowest term → $\frac{\overset{1}{\cancel{10 \div 10}}}{\underset{3}{\cancel{30 \div 10}}}$ =

$\frac{1}{3}$ ← reduced and in simplest form → $\frac{1}{3}$

How to Multiply Mixed Numbers

Example:

$(2\frac{1}{4}) \bullet (1\frac{1}{3})$ = ← rewrite as improper fractions

$\frac{9}{4} \bullet \frac{4}{3}$ = ← multiply numerators
multiply denominators

$\frac{36}{12}$ *or* 3 ← reduced and in simplest form

There is a shortcut called **canceling** that can be used when multiplying fractions. *Canceling* is dividing a numerator and a denominator by a *common factor* to write a fraction in lowest terms before multiplying fractions. You may use canceling to divide the numerator and the denominator of the same or different fractions.

Study the following two examples of canceling fractions.

How to Multiply Fractions

Example:

$\frac{3}{4} \bullet \frac{8}{9} =$ ← cancel by dividing the opposite numerator and denominator (by 3) to write in lowest terms → $\frac{\overset{1}{\cancel{3 \div 3}}}{4 \div 4} \bullet \frac{8 \div 4}{\underset{3}{\cancel{9 \div 3}}} =$

$\frac{\overset{1}{\cancel{3}}}{4} \bullet \frac{8}{\underset{3}{\cancel{9}}} =$ ← cancel by dividing the opposite numerator and denominator (by 4) to write in lowest terms → $\frac{1}{\underset{1}{\cancel{4 \div 4}}} \bullet \frac{\overset{2}{\cancel{8 \div 4}}}{3} =$

$\frac{1}{\underset{1}{\cancel{4}}} \bullet \frac{\overset{2}{\cancel{8}}}{3} =$ ← multiply numerators
multiply denominators →

$\frac{2}{3}$ ← reduced and in simplest form → $\frac{2}{3}$

How to Multiply Negative Mixed Numbers

Example:

$(-3\frac{1}{2})\ (4\frac{2}{3}) =$ ← rewrite mixed numbers as improper fractions

$\frac{-7}{2} \bullet \frac{14}{3} =$ ← cancel by dividing numerator and denominator (by 2) to write in lowest terms → $\frac{-7}{\underset{1}{\cancel{2 \div 2}}} \bullet \frac{\overset{7}{\cancel{14 \div 2}}}{3} =$

$\frac{-7}{\underset{1}{\cancel{2}}} \bullet \frac{\overset{7}{\cancel{14}}}{3} =$ ← multiply numerators
multiply denominators
(negative • **positive** = negative) →

$\frac{-49}{3}$ *or* $-16\frac{1}{3}$ ← reduced and in simplest form → $\frac{-49}{3}$ *or* $-16\frac{1}{3}$

Dividing Fractions

To divide fractions, multiply by the **reciprocal** of the *divisor*. Two numbers are *reciprocals* if their product is one.

Since

$$\frac{{}^{1}\cancel{3}}{{}_{1}\cancel{5}} \bullet \frac{\cancel{5}^{1}}{\cancel{3}_{1}} = 1,$$

$\frac{3}{5}$ is the reciprocal of $\frac{5}{3}$ and

$\frac{5}{3}$ is the reciprocal of $\frac{3}{5}$.

For a fraction not equal to 0, find its reciprocal by *inverting* or turning the fraction upside down, like $\frac{3}{5}$ and $\frac{5}{3}$. In everyday language you could say the following: *Flip* the *second* fraction and multiply.

Study the following three examples and see if you can see how the rule works.

How to Divide Fractions and Whole Number

Example:

$\frac{9}{10} \div 2 =$ ← remember, $2 = \frac{2}{1}$

$\frac{9}{10} \div \frac{2}{1} =$ ← invert the second fraction (reciprocal) and multiply

$\frac{9}{10} \bullet \frac{1}{2} =$ ← multiply numerator multiply denominator

$\frac{9}{20}$ ← reduced and in simplest form

How to Divide Fractions

Example:

$\frac{3}{4} \div \frac{2}{3} =$ ← invert the second fraction (reciprocal) and multiply

$\frac{3}{4} \bullet \frac{3}{2} =$ ← multiply numerator multiply denominator

$\frac{9}{8}$ *or* $1\frac{1}{8}$ ← reduced and in simplest form

How to Divide Mixed Numbers and Fractions

Example:

$2\frac{2}{5} \div \frac{8}{15} =$

change mixed numbers to improper fraction

$\frac{12}{5} \div \frac{8}{15} =$

invert second fraction and multiply

$\frac{12}{5} \bullet \frac{15}{8} =$

cancel by dividing the opposite numerator and denominator (by 4) to write in lowest terms

$\frac{{}^{3}\cancel{12}}{5} \bullet \frac{15}{\cancel{8}_{2}} =$

$\frac{{}^{3}\cancel{12 \div 4}}{5 \div 5} \bullet \frac{15 \div 5}{\cancel{8 \div 4}_{2}} =$

cancel by dividing the opposite numerator and denominator (by 5) to write in lowest terms

$\frac{3}{{}_{1}\cancel{5}} \bullet \frac{\cancel{15}^{3}}{2} =$

$\frac{3}{{}_{1}\cancel{5 \div 5}} \bullet \frac{\cancel{15 \div 5}^{3}}{2} =$

multiply numerator
multiply denominator

$\frac{9}{2}$ *or* $4\frac{1}{2}$

reduced and in simplest form

$\frac{9}{2}$ *or* $4\frac{1}{2}$

How to Solve Equations by Mulitplying Using Reciprocals

Example:

$\frac{3}{4}x = \frac{7}{8}$

multiply both sides by the reciprocal of $\frac{3}{4}$

${}^{1}\frac{\cancel{4}}{3} \bullet \frac{3}{\cancel{4}_{1}}\, x = {}^{1}\frac{\cancel{4}}{3} \bullet \frac{7}{\cancel{8}_{2}}$

cancel by dividing the opposite numerator and denominator (by 4) to write in lowest terms

${}^{1}\frac{\cancel{4 \div 4}}{3} \bullet \frac{3}{\cancel{4 \div 4}_{1}}\, x = {}^{1}\frac{\cancel{4 \div 4}}{3} \bullet \frac{7}{\cancel{8 \div 4}_{2}}$

$\frac{1}{{}_{1}\cancel{3}} \bullet \frac{\cancel{3}^{1}}{1}\, x = \frac{1}{3} \bullet \frac{7}{2}$

cancel by dividing the opposite numerator and denominator (by 3) to write in lowest terms

$\frac{1}{{}_{1}\cancel{3 \div 3}} \bullet \frac{\cancel{3 \div 3}^{1}}{1}\, x = \frac{1}{3} \bullet \frac{7}{2}$

multiply numerator
multiply denominator

$x = \frac{7}{6}$ *or* $1\frac{1}{6}$

reduced and in simplest form

$x = \frac{7}{6}$ *or* $1\frac{1}{6}$

Practice

Answer the following. Write each answer in **simplest form**. *Show all your work.*

1. $\frac{9}{2} \bullet \frac{4}{3} =$

2. $\frac{5}{2} \bullet \frac{2}{5} =$

3. $\frac{3}{4} \bullet 3 =$

4. $\frac{1}{3} \bullet \frac{3}{4} =$

5. $\frac{5}{8} \bullet \frac{4}{5} =$

6. $\frac{3}{2} \bullet \frac{1}{4} =$

✓ **Check yourself:** *Use the scrambled answers below and check your answers to problems 1-6 above.*

$\frac{1}{4}$ $\frac{3}{8}$ $\frac{1}{2}$ 1 $2\frac{1}{4}$ 6

Answer the following. Write each answer in **simplest form**. *Show all your work.*

7. $\frac{-5}{8} \bullet \frac{2}{5} =$

9. $\frac{7}{10} \bullet \frac{-5}{4} =$

8. $\frac{-1}{4} \bullet \frac{-1}{4} =$

10. $3 \bullet \frac{3}{4} =$

Practice

Answer the following. Write each answer in **simplest form**. *Show all your work.*

1. $(2\frac{1}{2})(2\frac{1}{2}) \quad =$

5. $(1\frac{2}{3})(2\frac{1}{2}) \quad =$

2. $(1\frac{1}{2})(1\frac{1}{3}) \quad =$

6. $(-2\frac{3}{8})\,(-4) \quad =$

3. $(2\frac{1}{4})(3\frac{1}{2}) \quad =$

7. $(2\frac{1}{5})(-1\frac{1}{2}) \quad =$

4. $(3\frac{1}{2})(1\frac{3}{4}) \quad =$

✓ **Check yourself:** *Use the scrambled answers below and check your answers to problems 1-7 above.*

$-3\frac{3}{10} \quad 2 \quad 4\frac{1}{6} \quad 6\frac{1}{8} \quad 6\frac{1}{4} \quad 7\frac{7}{8} \quad 9\frac{1}{2}$

Answer the following. Write each answer in **simplest form**. *Show all your work.*

8. $\frac{2}{9} \div \frac{1}{3} \quad =$

11. $\frac{3}{4} \div \frac{2}{3} \quad =$

9. $\frac{3}{8} \div \frac{2}{3} \quad =$

12. $\frac{2}{3} \div 4 \quad =$

10. $\frac{7}{5} \div \frac{7}{5} \quad =$

✓ **Check yourself:** *Use the scrambled answers below and check your answers to problems 8-12 above.*

$\frac{1}{6} \quad \frac{9}{16} \quad \frac{2}{3} \quad 1 \quad 1\frac{1}{8}$

Practice

Answer the following. Write each answer in **simplest form**. *Show all your work.*

1. $5\frac{1}{2} \div 1\frac{1}{3} \quad =$

4. $5 \div 2\frac{1}{3} \quad =$

2. $2\frac{1}{4} \div 1\frac{1}{4} \quad =$

5. $2\frac{1}{2} \div 2 \quad =$

3. $4\frac{1}{2} \div 2 \quad =$

Check yourself: *Use the scrambled answers below and check your answers to problems 1-5 above.*

$\mathbf{1\frac{1}{4}} \qquad \mathbf{1\frac{4}{5}} \qquad \mathbf{2\frac{1}{7}} \qquad \mathbf{2\frac{1}{4}} \qquad \mathbf{4\frac{1}{8}}$

Solve these equations. *Write each answer in* **simplest form**. *Show all steps.* **Refer** *to page 160 as needed.*

6. $\frac{2}{5}x = \frac{8}{25}$

7. $\frac{3}{4}x = 1\frac{1}{2}$

8. Find the *value* of $(\frac{3}{4})^2$.

Practice

Match each definition with the correct term. Write the letter on the line provided.

______ 1. a mathematical sentence that equates one expression to another expression

______ 2. any value for a variable that makes an equation or inequality a true statement

______ 3. to replace a variable with a numeral

______ 4. the result of an addition

______ 5. the result of a subtraction

______ 6. to find all numbers that make an equation or inequality true

______ 7. numbers less than zero

A. difference

B. equation

C. negative numbers

D. solution

E. solve

F. substitute

G. sum

Practice

Circle the letter of the correct answer.

1. When ____________ , you divide a numerator and a denominator by a common factor to write a fraction in lowest terms.
   a. canceling
   b. substituting
   c. placing

2. Two numbers whose product is 1 are called ____________ .
   a. unit rates
   b. variables
   c. reciprocals

3. A number and its opposite whose sum is zero (0) are ____________ .
   a. additive inverses
   b. terminating decimals
   c. substitutes

4. The length of the boundary around a figure or the distance around a polygon is called the ____________ .
   a. sum
   b. perimeter (P)
   c. simplest form

5. 0.75 and 75% are ____________ because they are the same number expressed in different forms.
   a. equivalent
   b. remainders
   c. factors

6. LCD is an abbreviation for ____________ .
   a. longest cross division
   b. length common decimal
   c. least common denominator

7. Numbers greater than zero are ____________ .
   a. negative numbers
   b. common factors
   c. positive numbers

Adding and Subtracting Decimals

The following is important to remember when we add or subtract decimals:

- Write the decimals so that the decimals line up vertically (↕).
- Write whole numbers as decimal numbers by adding a decimal point and zeros.
- Place the decimal point in the sum or difference so that it lines up vertically with the decimal points in the problem.
- Add or subtract as with whole numbers.
- Bring the decimal point straight down into the sum or difference.

Add: 3.29 + 1.34	$\begin{array}{r} 3.29 \\ +\ \underline{1.34} \\ 4.63 \end{array}$	Line up decimals vertically.
Add: 25 + 3.06	$\begin{array}{r} 25.00 \\ +\underline{\ 3.06} \\ 28.06 \end{array}$	Write the decimal point and two zeros. Line up decimals vertically.
Subtract: 8.8 – 2.5	$\begin{array}{r} 8.8 \\ -\ \underline{2.5} \\ 6.3 \end{array}$	Line up decimals vertically.
Subtract: 25 – 3.8	$\begin{array}{r} 25.0 \\ -\underline{\ \ 3.8} \\ 21.2 \end{array}$	Write the decimal point and one zero. Line up decimals vertically.

Practice

Add *or* subtract *as indicated. Show all work.*

1. 8.3 + 2.9 =

2. 13 + 9.31 =

3. 8.3 + 6.9 + 8.4 =

4. 20.64 + 5.6 + 4.28 =

5. 32 + 5.4 + 2.6 =

6. 5.43 – 0.18 =

7. 38.8 – 11.4 =

10. 24 – 15.6 =

8. 16 – 6.8 =

11. 9.06 – 4.44 =

9. 12.4 – 7 =

12. 78 – 72.75 =

Practice

Use the examples below and the methods learned in Unit 1 to solve **one-step equations involving decimals**.

Example:

$$\begin{aligned} x + 2.8 &= 12 && \text{subtract 2.8 from both sides} \\ x + 2.8 - 2.8 &= 12 - 2.8 \\ x &= 9.2 \end{aligned}$$

Example:

$$\begin{aligned} y - 5.6 &= 8.5 && \text{add 5.6 to both sides} \\ y - 5.6 + 5.6 &= 8.5 + 5.6 \\ y &= 14.1 \end{aligned}$$

Solve these equations. *Show all steps.*

1. $a + 4.1 = 12$

2. $x - 4.1 = 12$

3. $y + 5.8 = 8.3$

4. $22.7 = x + 2.4$

5. $0 = n - 7.9$

6. $30 = y + 22.38$

7. $a - 13.1 = 4.082$

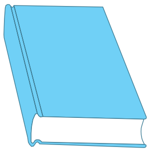
8. $d + 0.49 = 36$

Solve the following. Show all your work.

9. Find Jan's total monthly cost of owning and maintaining an automobile if her expenses are as follows:

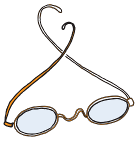
Monthly car payment:	$375.50
Monthly insurance:	$85.00
Average cost of gasoline per month:	$65.00
Average maintenance cost per month:	$22.50

Answer: ____________



10. Joe bought a book for $34.58. If he paid with two $20 bills, what was his change?

Answer: ____________



11. Ian's new eyeglasses costs a total of $385.89. The frames of the glasses are $129.95. Find the cost of the lenses.

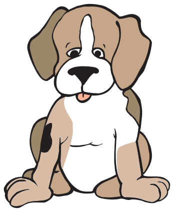
Answer: ____________

12. Aunt Nellie's gardener plans to install a fence around her *triangular*-shaped flower garden. The sides measure 12.5 meters, 10.8 meters, and 8.3 meters. Calculate the amount of fence material needed.

Answer: ____________

13. Andy needs to build a fence for his new puppy. The space to be enclosed is a *rectangular* shape with a length of 20.5 feet and a width of 12.5 feet. What is the perimeter of the rectangular shape?

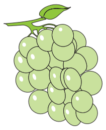
Answer: ____________

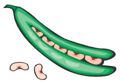


Multiplying and Dividing Decimals

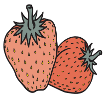
You may not think about it, but decimals are used a lot at a grocery store. For example, at the ABC Produce Market, fruits and vegetables are priced per pound.

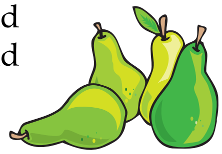
Use these typical prices per pound to work through the following examples:





strawberries	$1.25	per pound
bananas	$0.49	per pound
grapes	$0.99	per pound
pears	$1.15	per pound
pole beans	$1.29	per pound

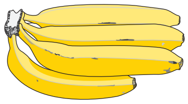




Example:

To find the cost of 4 pounds of bananas, we can *estimate* by rounding $0.49 to $0.50 and think:

4 x $0.50 is $2.00.



or

We can calculate the *exact cost* by multiplying 4 x $0.49. Then we count the number of digits to the right of the decimal point.

$0.49		$\boxed{2}$ digits to the right of the decimal point
x 4	+	$\boxed{0}$ digits to the right of the decimal point
$1.96		$\boxed{2}$ digits to the right of the decimal point

Then count off the same number of digits in the product.

Remember: Each place to the right of the decimal point is a *decimal place*. The number of decimal places in the product must equal the *sum* of the number of decimal places in the factors.

When multiplying decimals:

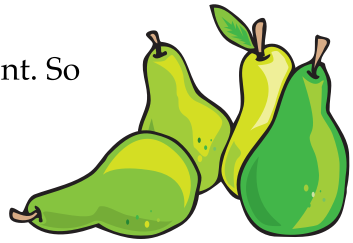
1. Multiply as if the factors were whole numbers.
2. Count the number of digits (decimal places) to the right of the decimal point in each factor and *add*.
3. Count off the *sum* of the digits (decimal places) in the factors (from right to left) and insert a decimal point. (Sometimes you will need to add zeros as place holders in the product.)

Example:

Suppose we buy 3.5 pounds of pears. Find the cost by multiplying 3.5 x $1.15.

$$\begin{array}{r l}
\$1.15 & \quad \boxed{2}\text{ digits to the right of the decimal point} \\
\text{x}\ \ 3.5 & +\ \boxed{1}\text{ digit to the right of the decimal point} \\
\hline
575 & \\
345 & \\
\hline
\$4.\underset{\curvearrowleft}{025} & \quad \boxed{3}\text{ digits to the right of the decimal point}
\end{array}$$

$4.025 will have to be *rounded* to the nearest cent. So our cost would be $4.03.



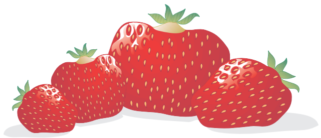
Example:

If Katie has $5.00, how many pounds of strawberries can she buy? Since each pound of strawberries is $1.25, we need to determine how many times $1.25 will divide into $5.00.

Remember: When dividing by decimals, you always want to divide by a whole number. So sometimes you must multiply the *dividend* and the *divisor* by a power of 10, moving the decimal point to the right. Then divide as if you were dividing by whole numbers.

$$\begin{array}{r}
4. \\
\underset{\curvearrowright}{1.25}\overline{)\underset{\curvearrowright}{5.00}} \\
\underline{500}
\end{array}$$

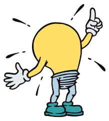
Move both decimal points 2 places to the *right* and then divide.



Katie can buy 4 pounds of strawberries.

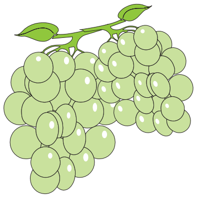
Notice that some items are priced at $ 0.49, $ 0.99, and $1.29 per pound. We can estimate the number of pounds we can buy with a certain amount of money to the nearest 10 cents.

Think: $0.49 .... $0.50
$0.99 .... $1.00
$1.29 .... $1.30



If we have $3.00, a quick way to determine how many pounds of grapes (at $0.99 a pound) we can buy is to divide $3.00 by $1.00.

$$\begin{array}{l} \qquad\quad\; 3.\text{ pounds of grapes} \\ 1.\underset{\curvearrowright}{00}\overline{)3.\underset{\curvearrowright}{00}} \\ \qquad\;\; \underline{300} \end{array}$$



Practice

Multiply ***or*** **divide** *as indicated. For* **division** *problems,* **if necessary**, *carry out the division to the* **nearest thousandth**. *Then* **round the quotient to the nearest hundredth**. *Show all of your work.*

1. 3.2 x 2 =
2. 6.5 x 1.2 =
3. 0.75 x 12 =
4. 5.7 x 0.42 =
5. 0.004 x 0.14 =
6. 27 x 1.4 =
7. 1.025 ÷ 5 =
8. 358.4 ÷ 56 =

9. 616.2 ÷ 6 =

13. 31.2 ÷ 0.04 =

10. 6.1 ÷ 7 =

14. 1.875 ÷ 0.25 =

11. 74.2 ÷ 30 =

15. 8.8 ÷ 76 =

12. 3.75 ÷ 0.5 =

Refer to the **ABC Produce Market prices** *on page 175 for problems 16-20.* **Round to the nearest cent.** *Show all your work.*

16. How much would 3.5 pounds of bananas cost?

Answer: ____________________

17. Calculate the cost of 5.5 pounds of pole beans.

Answer: ____________________

18. Jenny needs 10 pounds of grapes for a picnic. Is a $10 bill enough? If so, how much change should Jenny receive?

Answer: ____________________ change

19. Joe has $20. If he buys 10 pounds of pears, how many pounds of strawberries can he buy without spending more than $20?

Answer: ____________________ pounds

20. Estimate the cost of 3 pounds of bananas and 3 pounds of grapes. Would a $5 bill cover the cost?

Estimate: ____________________

Answer: ____________________

Lesson Two Purpose

- Understand the relative size of integers, rational numbers, and real numbers. (MA.A.1.4.2)
- Understand that numbers can be represented in a variety of equivalent forms, including integers, fractions, decimals, percents, scientific notation, and exponents. (MA.A.1.4.4)
- Understand and explain the effects of addition, subtraction, multiplication, and division on real numbers, including exponents, and appropriate inverse relationships. (MA.A.3.4.1)
- Select and justify alternative strategies, such as using properties of numbers, including inverse, that allow operational shortcuts for computational procedures in real-world or mathematical problems. (MA.A.3.4.2)
- Add, subtract, multiply, and divide real numbers, including exponents, using appropriate methods of computing, such as mental mathematics, paper and pencil, and calculator. (MA.A.3.4.3)
- Use estimation strategies in complex situations to predict results and to check the reasonableness of results. (MA.A.4.4.1)
- Use concrete and graphic models to derive formulas for finding rate, distance, and time. (MA.B.1.4.2)
- Relate the concepts of measurement to similarity and proportionality in real-world situations. (MA.B.1.4.3)
- Solve real-world problems involving rated measures (miles per hour, feet per second). (MA.B.2.4.2)

Writing Numbers in Scientific Notation

In Unit 1, we learned to use **exponents**. When you multiply 10s together, the product is called a *power of 10*. Powers of 10 are used to represent very large quantities and extremely small quantities. *Exponents* can be used to show a power of 10. The exponent tells the number of times that 10 is a factor.

Look at powers of 10 below and notice that we have extended our study to include negative exponents.

$10^4 = 10{,}000$
$10^3 = 1{,}000$
$10^2 = 100$
$10^1 = 10$
$10^0 = 1$
$10^{-1} = \frac{1}{10^1} = \frac{1}{10} = 0.1$
$10^{-2} = \frac{1}{10^2} = \frac{1}{100} = 0.01$
$10^{-3} = \frac{1}{10^3} = \frac{1}{1{,}000} = 0.001$
$10^{-4} = \frac{1}{10^4} = \frac{1}{10{,}000} = 0.0001$

Scientific notation is a *shorthand* method of writing very large or very small numbers using exponents. The number in *scientific notation* is expressed as a product of a power of 10 *and* a number that is greater than or equal to 1 and less than 10.

To change a number written in decimal notation to scientific notation, write it as a *product* of two *factors*:

(a decimal greater than or equal to 1 and less than 10) x (a power of 10)

Where *a* is a decimal ≥ 1 and < 10

$a \bullet 10^n$

A power of 10—where *n* is an integer {…, -2, -1, 0, 1, 2, …}

Example 1: speed of light = 300,000,000 miles per second, written in scientific notation = 3.0×10^8

Example 2: light travels 1 meter in 0.0000000033 seconds, written in scientific notation = 3.3×10^{-9}

Example:

Write 51,000,000 and 0.000865 in scientific notation.

Large Number	**Small Number**
51,000,000	0.000865

1. Count the number of places we move the decimal point to get a number between 1 and 10.

Remember: Only *one* nonzero digit (a number from 1 to 9) can be to the *left* of the decimal point.

51,000,000.	0.000865
7 places to the *left*	4 places to the *right*

2. Write the number as a *product* of a number between 1 and 10 and a power of 10.

$$\underbrace{5.1}_{\text{5.1 is a decimal} \geq 1 \text{ and} < 10.} \times \underbrace{10^{7}}_{\text{Power of 10 with positive exponent 7.}} \qquad \underbrace{8.65}_{\text{8.65 is a decimal} \geq 1 \text{ and} < 10.} \times \underbrace{10^{-4}}_{\text{Power of 10 with negative exponent -4.}}$$

Consider these large quantities:	In scientific notation:
700,000,000	7×10^{8}
980,000	9.8×10^{5}
40,000,000	4×10^{7}
250,000,000,000	2.5×10^{11}

And these very small quantities:	In scientific notation:
0.0085	8.5×10^{-3}
0.000009	9×10^{-6}
0.000000556	5.56×10^{-7}
0.0000302	3.02×10^{-5}

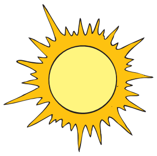
Practice

Write in **scientific notation**.

1. 22,000 ____________________
2. 500,000 ____________________
3. 0.00075 ____________________
4. 31,000 ____________________
5. 0.00382 ____________________
6. 0.00006 ____________________
7. 2,110,000 ____________________
8. 0.0000085 ____________________
9. 8,080,000 ____________________
10. 21,000 ____________________
11. 0.00021 ____________________
12. 900,000,000 ____________________

Notice in these word statements that the quantities are preceded by the word **about**. *That means that we are dealing with* **estimates** *rather than* **exact** *quantities.*

Write each number in scientific notation.



13. The sun is about 4,600,000,000 years old.

____

14. There are about 194,000,000 telephones in the United States.

____

15. The thickness of a human hair is about 0.00388 of a foot.

____

16. The length of an infrared light wave is about 0.0000037 of a meter.

____

17. The distance traveled by light in one year is called a light-year. One light year is about 6,000,000,000,000 miles.

____

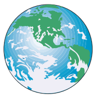
18. Our local bookstore sold about 23,000 books last month.

____

19. An influenza virus is about 0.00000012 of a meter in diameter.

____

20. The Earth is about 93,000,000 miles from the sun.



____

Ratios and Rates

The first question we ask is "What is the difference between a **ratio** and a **rate**?"

A *ratio* is a comparison of *two like quantities*. A ratio is actually a fraction which can be written in the following ways.

The ratio of 1 to 4 can be written

$$\left.\begin{array}{l} \frac{1}{4} \textit{ or} \\ 1 \text{ to } 4 \textit{ or} \\ 1{:}4. \end{array}\right\} \text{Each one of these is read as } \textit{one to four}.$$

When writing ratios, *order* is important. All of these expressions are read *one to four*. If the *colon notation* is used, like in the last example above, the first number is divided by the second.

The ratio of 5 to 11 is

$\frac{5}{11}$, not $\frac{11}{5}$ *or*

5:11, not 11:5.

Also, we usually express ratios in *lowest terms*. A ratio is said to be in lowest terms if the two numbers are relatively *prime*. You do *not* change an improper fraction to a mixed number if the improper fraction is expressed as a ratio. For example:

The ratio of $25 to $15 is written

$$\frac{\$25}{\$15} = \frac{25 \div 5}{15 \div 5} = \frac{5}{3}.$$

Note: We leave *ratios* as *improper fractions* in lowest terms and do *not* rewrite as mixed numbers.

Rates are used to compare *different kinds of quantities*.

Note: We write *rates* as *fractions in simplest form*.

Example 1:

Suppose Jake ran 4 miles in 44 minutes. Find Jake's rate (or average speed).

As a rate, we write Jake's distance (4 miles) over his time (11 minutes) as a fraction. Then we reduce and write in simplest form.

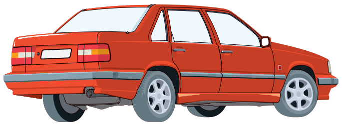
$$\text{Rate} = \frac{\text{distance}}{\text{time}} = \frac{4\text{ miles}}{44\text{ minutes}} = \frac{1\text{ mile}}{11\text{ minutes}} \textit{ or } 1\text{ mile per 11 minutes}$$

(Always write the units—or fixed quantity—in your answer.)

So, Jake's average rate of speed was 1 mile in 11 minutes. In reflection, this answer is *reasonable* because if Jake's average speed was 11 minutes per mile and he traveled 4 miles, it would have taken him 44 minutes.

Example 2:

You drive 300 miles in 5 hours. Find your rate (or average speed). Use the same formula of rate equals distance over time. Write in your distance (300 miles) over your time (5 hours) and reduce the fraction to simplest form.

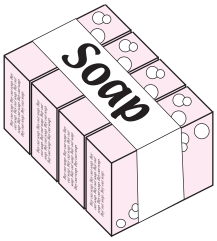


$$\text{Rate} = \frac{\text{distance}}{\text{time}} = \frac{300\text{ miles}}{5\text{ hours}} = \frac{60\text{ miles}}{1\text{ hour}} \textit{ or } 60\text{ miles per hour}$$

So your average rate of speed was 60 miles per hour. In reflection, this answer is *reasonable* because if your average rate of speed was 60 miles per hour and you traveled for 5 hours, you would have gone 300 miles.

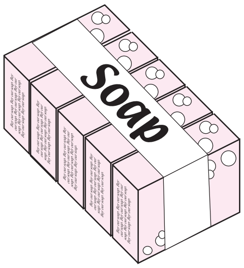
We actually found a **unit rate** since we read 60 mi/hr as 60 miles *per* hour. A *unit rate* is a rate for one **unit** of a given quantity. A unit rate has a denominator of 1.

Unit rates are used a lot in **unit pricing**. A *unit price* is the cost of a particular item, expressed in the unit in which the product is generally measured. A unit price enables us to find the *better buy*. For example, which package of soap below is the better buy?



4 bars of soap

$0.99



5 bars of soap

$1.15

First, determine the price per bar.

4 bars: $\frac{\$0.99 \div 4}{4 \div 4} = \frac{\$0.25}{1 \text{ bar}}$ *or* $0.25 per bar

Answer is reasonable because 4 bars at $0.25 equals $1.00. (The $0.25 was rounded to the nearest cent, that is why 4 bars did not equal $0.99.)

5 bars: $\frac{\$1.15 \div 5}{5 \div 5} = \frac{\$0.23}{1 \text{ bar}}$ *or* $0.23 per bar

Answer is reasonable because 5 bars at $0.23 equals $1.15.

The package containing 5 bars of soap is the better buy. Each bar of soap costs $0.23, which is less than each bar in the 4 bar-package. But be careful—buying a package with more items in the package will *not* always result in the better buy. You must *do the math* to be sure.

Practice

Write each **ratio** *as a fraction in* **lowest terms**.

Remember: Leave *ratios* as *improper fractions* written in *lowest terms*; do **not** rewrite in simplest form as mixed numbers.

1. 4 to 8 = ____________
2. 18 to 12 = ____________
3. 8 to 6 = ____________
4. 16 to 36 = ____________
5. 12:20 = ____________
6. 24 to 18 = ____________
7. 10:4 = ____________
8. $\frac{24}{96}$ = ____________
9. $\frac{9}{45}$ = ____________
10. 14 centimeters to 20 centimeters = ____________
11. 10 inches to 12 inches = ____________
12. $32 to $100 = ____________

Write each **rate** *as a fraction in* **simplest form**.

Remember: Write *rates* as *fractions* in *simplest form*.

13. 8 phone lines for 36 employees ______________________________
14. 4 inches of rain in 18 hours ______________________________
15. 8 flight attendants for 200 passengers ______________________________
16. 210 pounds of grass seed for 9 lawns ______________________________
17. 6 shrubs every 18 feet ______________________________
18. 8 laser printers for 28 computers ______________________________

Write each **rate** *as a* **unit rate**.

19. 72 diapers for 12 babies ______________________________

20. 200,000 library books for 4,000 students ___________________

21. 750 riders in 5 subway cars ___________________________

Find the **unit rate**. *Check for reasonableness of results. Show all your work.*

Remember: rate $= \dfrac{\text{distance}}{\text{time}}$

22. Dathan drove 225 miles in 5 hours. What was Dathan's rate (or average speed)?

    Answer: __________ miles per hour

23. Jenna reached her goal of running 3 miles in 30 minutes. What was Jenna's rate (or average speed)? **Hint:** 60 minutes = 1 hour.

    Answer: __________ miles per hour

Find each **unit price** *and decide which is the* **better buy**. *Check for reasonableness of results.* **Round to the nearest thousandth** *(three decimal places). Show all your work.*

24. Steak sauce at $2.29 for 12 ounces *or* $1.49 for 8 ounces

$2.29 for 12 ounces = ____________ per ounce

$1.49 for 8 ounces = ____________ per ounce

Better buy: ____________ ounces of steak sauce

25. Olives at $1.89 for 32 ounces *or* $0.89 for 18 ounces

$1.89 for 32 ounces = ____________ per ounce

$0.89 for 18 ounces = ____________ per ounce

Better buy: ____________ ounces of olives

26. Napkins at 100 for $0.59 *or* 180 for $0.93

100 for $0.59 = ____________ each

180 for $0.93 = ____________ each

Better buy: package of ____________ napkins

Writing and Solving Proportions

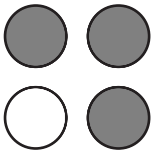
An equation showing that two ratios (or *rates*) are equal is called a **proportion**.

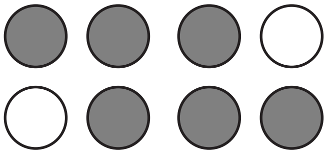
$\frac{3}{4} = \frac{6}{8}$ is a *proportion*.

We read this as *3 is to 4 as 6 is to 8*.

Also, we can see that the proportion $\frac{3}{4} = \frac{6}{8}$ is *true*.

Look below. In the first set of circles, 3 out of 4 circles are shaded. In the second set, 6 out of 8 circles are shaded. And although the second set has more circles, the ratio of shaded circles to total circles is the same. That is, $\frac{3}{4} = \frac{6}{8}$. One ratio is *equal* to the other ratio, and is therefore called a *proportion*.





The ability to compare and produce equal ratios involves *proportional reasoning*. A common use of proportions is to make or use maps and scale models.

There are two ways to find out whether a proportion is true.

- One method is to write each fraction in simplest form and compare them.

  Since $\frac{6}{8}$ simplifies to $\frac{3}{4}$, we note that $\frac{3}{4} = \frac{6}{8}$ is true.

- Another way to determine if the proportion is true is to use *cross products* with **cross multiplication**. *Cross multiplication* is a method for solving and checking proportions.

$$\frac{3}{4} = \frac{6}{8}$$

$8(3) = 4(6)$ When cross products are *equal*, the proportion *is true*.

$24 = 24$

If cross products are *not equal*, the proportion is *false*.

We ask the following question.

Is $\frac{2}{3} = \frac{4}{9}$ a proportion?

$$\frac{2}{3} \times \frac{4}{9}$$

Is $9(2) = 3(4)$?

Does $18 = 12$? Of course not!

Therefore, $\frac{2}{3} = \frac{4}{9}$ is *not* a proportion.

Examples:

We use cross products to state whether each proportion is true or false.

a. $\frac{2}{11} = \frac{1}{5}$

Is $5(2) = 11(1)$?

10 is *not* equal to 11
Answer: *false*

b. $\frac{2}{3} = \frac{5}{6}$

Is $6(2) = 3(5)$?

12 is *not* equal to 15
Answer: *false*

c. $\frac{1}{2} = \frac{3}{6}$

Is $6(1) = 2(3)$?

6 is equal to 6
Answer: *true*

As we've already seen, a proportion states that two ratios are equal. Frequently, proportions are presented as follows:

$\frac{a}{b} = \frac{c}{d}$ In words: *a* is to *b* as *c* is to *d*.

Also:

$$d(a) = b(c)$$

To solve a proportion when three parts are known, we use cross multiplication or the *cross product property*.

Solve:

$\frac{x}{3} = \frac{1}{6}$ ← use cross product property

$6(x) = 3(1)$

$\frac{6x}{6} = \frac{3}{6}$ ← divide each side by 6

$x = \frac{1}{2}$ ← reduced and in simplest form

We can check our solution by substituting the $\frac{1}{2}$ in the original proportion.

Is $\frac{\frac{1}{2}}{3} = \frac{1}{6}$?

Is $\frac{1}{2}(6) = 3(1)$?

Is $(3) = 3$?

Yes, our solution is correct.

or

Since $\frac{1}{2} = 0.5$

Is $\frac{0.5}{3} = \frac{1}{6}$?

Is $0.5(6) = 3(1)$?

Is $3.0 = 3$?

Yes, our solution is correct.

The cross product property *only* works when solving a proportion. It does not apply to doing operations with fractions, such as multiplying or dividing.

Practice

Write **True** *if the proportion is equal. Write* **False** *if the proportion is* not *equal.*

__________ 1. $\frac{15}{9} = \frac{5}{3}$

__________ 2. $\frac{7}{12} = \frac{4}{7}$

__________ 3. $\frac{7}{8} = \frac{5}{6}$

__________ 4. $\frac{2}{1} = \frac{16}{8}$

Write each sentence as a **proportion***. Then solve.*

5. 8 is to 6 as 4 is to *n*.

6. 12 is to *n* as 2 is to 3.

7. *n* is to 18 as 4 is to 9.

8. 8 is to 15 as *n* is to 6.

Solve each **proportion**.

9. $\frac{30}{10} = \frac{15}{n}$

11. $\frac{2}{1} = \frac{n}{3.5}$

10. $\frac{2}{x} = \frac{15}{9}$

12. $\frac{n}{3} = \frac{18}{54}$

Check yourself. Use a calculator to check problems 9-12.

For example,

$$\frac{35}{10} = \frac{n}{7}$$

you would enter

35 $\boxed{\text{x}}$ 7 $\boxed{\div}$ 10 $\boxed{=}$

and get the answer 24.5.

Solve each **proportion**.

13. $\frac{y}{0.6} = \frac{0.05}{12}$

15. $\frac{8}{\frac{1}{3}} = \frac{24}{n}$

14. $\frac{0.2}{0.7} = \frac{8}{n}$

16. $\frac{n}{48} = \frac{\frac{3}{4}}{12}$

Practice

Write **True** *if the statement is correct. Write* **False** *if the statement is* not *correct.*

__________ 1. *Scientific notation* is the longhand method of writing numbers without exponents.

__________ 2. The *ratio* of 1 to 4 can be written $\frac{1}{4}$ *or* 1 to 4 *or* 1:4.

__________ 3. When writing ratios, order is *not* important.

__________ 4. Leave ratios as improper fractions and do *not* rewrite as mixed numbers.

__________ 5. A *unit* is an estimated or approximate quantity used to measure.

__________ 6. *Rates* are used to compare different kinds of quantities.

__________ 7. *Unit rates* are used a lot in unit pricing, which enables us to find the *better buy*.

__________ 8. *Unit pricing* is the cost of a particular item, such as one bar of soap.

__________ 9. An equation showing that two ratios (or *rates*) are equal is called a *proportion*.

_________ 10. One way to determine if the proportion is true or false is to use *cross products* with cross multiplication.

_________ 11. If cross products are *not* equal, the proportion is true.

Using Proportions

Writing proportions is one of the most important skills we can learn. Courses in science, business, health, engineering and, of course, mathematics, require a solid foundation in handling proportions. Additionally, we use proportions in everyday situations.

When we are given a specified ratio (or rate) of two quantities, a proportion can be used to find an unknown quantity.

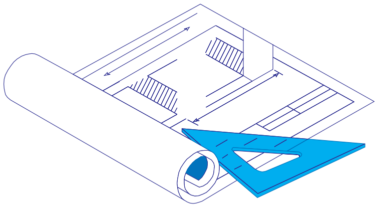
Example 1:

On an architect's blueprint, 1 inch corresponds to 6 feet. How long is a deck represented by a $3\frac{1}{2}$ inch line on the blueprint?

$$\frac{1 \text{ inch}}{6 \text{ feet}} = \frac{3\frac{1}{2} \text{ inches}}{x \text{ feet}}$$

Note: If the *left* side of our proportion compares blueprint *length to actual length*, then the *right* side must also compare blueprint *length to actual deck length*.

Use cross products:



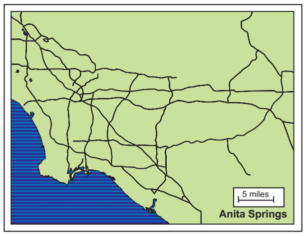
$$\begin{aligned} \frac{1}{6} &= \frac{3\frac{1}{2}}{x} \\ 1x &= 6(3\tfrac{1}{2}) \\ x &= 21 \text{ feet} \end{aligned} \qquad \textit{or} \qquad \begin{aligned} \frac{1}{6} &= \frac{3.5}{x} \\ 1x &= 6(3.5) \\ x &= 21 \text{ feet} \end{aligned}$$

Therefore, 21 feet is represented by $3\,\frac{1}{2}$ *or* 3.5 inches on the blueprint.

Example 2:

On a map of Anita Springs, 5 miles corresponds to 2 inches. How many miles correspond to 7 inches?

Translate: Let *n* represent our unknown miles. Since 5 miles corresponds to 2 inches as *n* miles corresponds to 7 inches, we can write the proportion:



$$\begin{aligned}
\begin{array}{l}\text{miles}\\ \text{inches}\end{array}\ \frac{5}{2} &= \frac{n}{7}\ \begin{array}{l}\text{miles}\\ \text{inches}\end{array}\\
5(7) &= 2n\\
35 &= 2n\\
17\tfrac{1}{2} &= n \qquad \textit{or} \qquad 17.5 = n
\end{aligned}$$

So, 17 $\frac{1}{2}$ *or* 17.5 miles corresponds to 7 inches on the map.

Example 3:

The instructions on a bottle of cough syrup state that the patient should take $\frac{1}{2}$ teaspoon for every 40 pounds (lbs) of body weight. At this rate, find the appropriate dose for a 120-pound woman.

$$\begin{array}{rcl c rcl}
\begin{array}{r}\text{teaspoons}\\ \text{pounds}\end{array}\ \frac{\frac{1}{2}}{40} &=& \frac{x}{120}\ \begin{array}{l}\text{teaspoons}\\ \text{pounds}\end{array} & & \frac{0.5}{40} &=& \frac{x}{120}\\
\frac{1}{2}(120) &=& 40x & \textit{or} & 0.5(120) &=& 40x\\
60 &=& 40x & & 60 &=& 40x\\
1\frac{1}{2} &=& x & & 1.5 &=& x
\end{array}$$

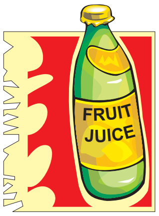
Thus, the 120-pound woman needs $1\frac{1}{2}$ *or* 1.5 teaspoons for each dose of cough syrup.

Practice

For each problem, set up a **proportion** *and* **solve**. *Check for reasonableness of results. Show all your work.*

1. If it takes Sandy 30 minutes to type and spell-check 4 pages on her computer, how long will it take her to type and spell-check 20 pages?

   Answer: ____________ minutes



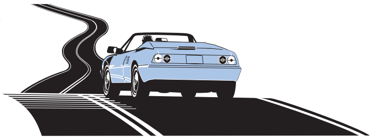
2. If a 2-liter bottle of fruit juice costs $3.98, how much would 6 liters cost?

   Answer: ____________

3. Archie's car averages 450 miles on a 12-gallon tank of gas. How many gallons of gas can Archie expect to use on a 2,025 mile trip?

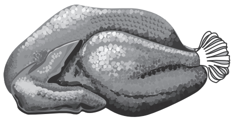
   Answer: ____________ gallons

4. After driving 7 hours, Jose had traveled 385 miles. If he continues to travel at the same rate, what distance will he have driven in 10 hours?



Answer: ____________ miles

5. If a 12-pound turkey takes 4 hours to cook, how long will it take to cook an 18-pound turkey?

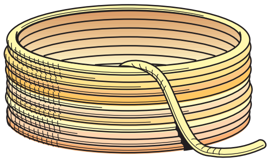


Answer: ____________ hours

6. The standard dose of an antibiotic is 6 cc (cubic centimeters) for every 50 pounds of body weight. At this rate, find the standard dose for a 175-pound woman.

Answer: ____________ cc

7. If 80 feet of rope weigh 20 pounds, how much would 100 feet of the same kind of rope weigh?



Answer: ____________ pounds

8. On an architect's blueprint, 1 inch corresponds to 8 feet. Find the length of a wall represented by a line $1\frac{7}{8}$ inches long on the blueprint.

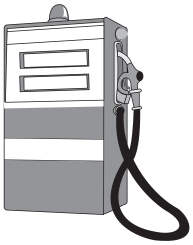
Answer: ____________ feet

9. A photograph measures a width of 4 inches by a length of 6 inches. If the photograph is enlarged so that the width is 6 inches, what is the length of the enlarged photograph?

Answer: ____________ inches

10. Paul spent $18 for 12 gallons of gasoline. At that price, how many gallons could he buy for $15?

Answer: ____________ gallons



Lesson Three Purpose

- Understand the relative size of integers, rational numbers, and real numbers. (MA.A.1.4.2)
- Understand concrete and symbolic representations of real numbers in real-world situations. (MA.A.1.4.3)
- Understand that numbers can be represented in a variety of equivalent forms, including integers, fractions, decimals, percents, scientific notation, and exponents. (MA.A.1.4.4)
- Select and justify alternative strategies, such as using properties of numbers, including inverse, that allow operational shortcuts for computational procedures in real-world or mathematical problems. (MA.A.3.4.2)
- Add, subtract, multiply, and divide real numbers, including exponents, using appropriate methods of computing, such as mental mathematics, paper and pencil, and calculator. (MA.A.3.4.3)

Percents

In previous sections, we reviewed fractions and decimals.

- To change a *fraction to a decimal*, we divide the denominator into the numerator.

Note: When setting up the division problem for the fraction $\frac{2}{5}$, it is helpful to remember that the fraction with its fraction bar reads *2 divided by 5*.

$\frac{2}{5}$ $\qquad$ $\begin{array}{r} 0.4 \\ 5\overline{)2.0} \end{array}$ $\quad$ so $\frac{2}{5} = 0.4$

fraction $\qquad$ decimal

$\frac{1}{2}$ $\qquad$ $\begin{array}{r} 0.5 \\ 2\overline{)1.0} \end{array}$ $\quad$ so $\frac{1}{2} = 0.5$

fraction $\qquad$ decimal

- To change a *decimal to a fraction*, simply count the decimal places and use the same number of zeros in the denominator.

$0.4 = \frac{4}{10} = \frac{2}{5}$

one decimal place $\qquad$ one zero

$0.002 = \frac{2}{1{,}000} = \frac{1}{500}$

three decimal places $\qquad$ three zeros

We know that we can express fractions as decimals, and decimals as fractions. We can also express fractions and decimals as **percents**. *Percent* literally means *per 100*, *hundredths*, or *out of every hundred*. The symbol % means *percent.* Its use dates back to the 1400s.

Look at the percents below. Each was changed from a percent to a fraction and then to a decimal.

$$5\% = \frac{5}{100} = 0.05 \qquad 6.2\% = \frac{6.2}{100} = \frac{62}{1{,}000} = 0.062$$

$$16\% = \frac{16}{100} = 0.16 \qquad 5\tfrac{1}{4}\% = 5.25\% = \frac{5.25}{100} = \frac{525}{10{,}000} = 0.0525$$

$$200\% = \frac{2\cancel{0}\cancel{0}}{1\cancel{0}\cancel{0}} = 2$$

Look for a pattern! Note how the decimal point *shifts* places when we change a percent to a decimal.

Rule: To change a *percent to a decimal*,

- drop the percent sign,
- move the decimal point *two places* to the *left*, and
- insert zeros as placeholders if necessary.

Remember: If you do *not* see a decimal point, it is at the end of the number.

Examples:

$$17.\cancel{\%} = 0.17$$

$$175.\cancel{\%} = 1.75$$

$$10.5\cancel{\%} = 0.105$$

$$30\tfrac{1}{2}\% = 30.5\cancel{\%} = 0.305$$

$$2\tfrac{3}{4}\% = 2.75\cancel{\%} = 0.0275$$

Rule: To change a *decimal to a percent*,

- move the decimal point *two places* to the *right*,
- insert zeros as placeholders as necessary, and
- then write the % sign.

Examples:

$0.\underset{\curvearrowright}{03} = 3\%$

$0.\underset{\curvearrowright}{37}5 = 37.5\%$

$0.\underset{\curvearrowright}{66}\tfrac{2}{3} = 66\tfrac{2}{3}\%$

$7.\underset{\curvearrowright}{} = 700\%$

Memory Trick: Here is an easy way to remember the two previous rules:

Decimal starts with a *D*.
Percent starts with a *P*.
D comes before P in the alphabet.

$D \xrightarrow{\text{right}} P$ (To move from *D* to *P* in the alphabet, move to the right)

To change a *decimal to a percent*, move the decimal point to the *right*

$0.\underset{\curvearrowright}{15} = 15\%$

$D \xleftarrow{\text{left}} P$ (To move from *P* to *D* in the alphabet, move to the left)

To change a *percent to a decimal*, move the decimal point to the *left*.

$\underset{\curvearrowleft}{85}\% = 0.85$

Practice

Write as **decimals**.

1. 58% = ____________

2. 65.3% = ____________

3. $42\frac{1}{4}\%$ = ____________

4. 180% = ____________

5. 500% = ____________

6. 4% = ____________

7. 0.1% = ____________

8. $5\frac{1}{2}\%$ = ____________

9. $6\frac{3}{4}\%$ = ____________

10. 6.3% = ____________

Write as **percents**.

11. 0.77 = ____________

12. 0.02 = ____________

13. 0.005 = ____________

14. $0.66\frac{1}{2}$ = ____________

15. 0.046 = ____________

16. 0.1 = ____________

17. 8 = ____________

18. 1 = ____________

19. $\frac{3}{5}$ = ____________

20. $\frac{1}{8}$ = ____________

Practice

Solve each problem. Show all your work.

Example:

John answered 15 problems out of 20 on his last math test. What percent of the problems did he answer correctly?

First, make a fraction: $\frac{\text{part}}{\text{total}}$ $\frac{15}{20}$

Reduce, by dividing both denominator and numerator by 5.

$\frac{15}{20} = \frac{3}{4}$

Second, change this fraction to a decimal: $\frac{3}{4}$ $\begin{array}{r} 0.75 \\ 4\overline{)3.00} \end{array}$

Third, change the decimal to a percent: $0.75 = 75\%$

1. Jared has been at bat 48 times and has hit 20 times. What percent of the time does he make a hit? **Round to the nearest tenth of a percent.**

   Answer: ____________

2. The down payment on a television is $150. The total price of the television is $750. The down payment is what percent of the cost of the television?

   Answer: ____________

3. Janice correctly spelled 20 words on a spelling test. There were 20 words on the test. What percent of the words did she spell correctly?

   Answer: ______________

4. The postman delivered mail to 171 out of the 180 houses on Florida Street. What percent of the houses did *not* get a mail delivery?

   Answer: ______________

5. 

   a. What percent of this rectangle is shaded? ____________________

   b. What percent of this rectangle is not shaded? ______________

   c. What is the sum of these two percents? ____________________

6. Which is *larger*, 15% or 0.145? Explain.

   Answer: ______________

   Explanation: ____

   ____

   ____

7. Which is *larger*, 200% or 5? Explain.

Answer: ____________

Explanation: ____

____

____

8. In a survey of 3,000 people, 2,400 said that they regularly eat breakfast.

What percent of the people surveyed eat breakfast? ____________

a. 8%

b. 85%

c. $87\frac{1}{2}\%$

d. 80%

9. There are certain fractions that appear very frequently in our lives that we need to memorize the relationships. Fill in the chart below and memorize it!

Fraction	**Decimal**	**Percent**
$\frac{1}{4}$		
$\frac{1}{2}$		
$\frac{3}{4}$		
$\frac{1}{5}$		
$\frac{2}{5}$		
$\frac{3}{5}$		
$\frac{4}{5}$		
$\frac{1}{8}$		
$\frac{3}{8}$		
$\frac{5}{8}$		
$\frac{7}{8}$		
$\frac{1}{3}$		
$\frac{2}{3}$		

Continued on following page.

Fraction	Decimal	Percent
1	______________	______________
$\frac{1}{10}$	______________	______________
$\frac{2}{10}$	______________	______________
$\frac{3}{10}$	______________	______________
0	______________	______________
$\frac{1}{6}$	______________	______________
$\frac{5}{6}$	______________	______________

Solving Percent Problems with Equations

We will see that many problems involving percents require that we take some *percent of a quantity*.

To do these problems, use the *fundamental rule for percentage*.

$$PB = A$$

$$\boxed{\begin{array}{l}\text{percent of base} = \text{ amount}\\[1em] \qquad\qquad PB \ = \ A\end{array}}$$

Note: P is the *percent*,
B is the *base*, and
A is the *amount*.

Example 1: Solve for the *amount* (A).

$$\underbrace{\text{5 percent}}\ \underset{\downarrow}{\text{of}}\ \underset{\downarrow}{20}\ \underset{\downarrow}{\text{is}}\ \underbrace{\text{what number?}}$$

$$\begin{array}{llcl}
\text{Equation:} & 5\%\ \text{x}\ 20 & = & A \longleftarrow \text{change 5\% to a decimal and multiply}\\
 & 0.05(20) & = & A\\
\text{Answer:} & 1 & = & A \longleftarrow \text{So: 5\% of 20 is 1.}
\end{array}$$

Example 2: Solve for the *base*.

$$\underbrace{\text{5 percent}}\ \underset{\downarrow}{\text{of}}\ \underbrace{\text{what number}}\ \underset{\downarrow}{\text{is}}\ \underset{\downarrow}{20?}$$

$$\begin{array}{llcl}
\text{Equation:} & 5\%\ \text{x}\ B & = & 20 \longleftarrow \text{change 5\% to a decimal and multiply}\\
 & 0.05B & = & 20\\
 & \frac{0.05B}{0.05} & = & \frac{0.20}{0.05} \longleftarrow \text{divide both sides by 0.05}\\
\text{Answer:} & B & = & 400 \longleftarrow \text{So: 5\% of 400 is 20.}
\end{array}$$

Example 3: Solve for the *percent*.

$$\underbrace{\text{What percent}}\ \underset{\swarrow}{\text{of}}\ \underset{\swarrow}{20}\ \underset{\downarrow}{\text{is}}\ \underset{\downarrow}{5}?$$

$$\begin{array}{llll}
\text{Equation:} & P(20) & = 5 & \leftarrow \text{use commutative property}\\
& & & \swarrow\\
& 20P & = 5 & \\
& \frac{20P}{20} & = \frac{5}{20} & \leftarrow \text{divide both sides by 20}\\
& P & = 0.25 & \leftarrow \text{change 0.25 to a percent for final answer}\\
& & & \swarrow\\
\text{Answer:} & P & = 25\% & \leftarrow \text{So: 25\% of 20 is 5.}
\end{array}$$

Notice: Since our answer needs to be a percent, we write 0.25 as 25%, our final answer.

Let's look at three more examples.

Example 4:

$$\underbrace{\text{What number}}\ \underset{\downarrow}{\text{is}}\ \underbrace{15\%\text{ of }60}?$$

$$\begin{array}{llll}
\text{Equation:} & A & = & PB\\
& A & = & 15\%(60)\\
& A & = & 0.15(60) \leftarrow \text{change 15\% to a decimal}\\
& A & = & 9 \quad \swarrow \quad \text{and multiply}\\
\text{Answer:} & \multicolumn{3}{l}{9 \text{ is } 15\% \text{ of } 60}
\end{array}$$

Example 5:

$$\underbrace{15\%\text{ of what number}}\ \underset{\downarrow}{\text{is}}\ \underset{\downarrow}{60}?$$

$$\begin{array}{llll}
\text{Equation:} & PB & = A & \\
& 15\%B & = A & \leftarrow \text{change 15\% to a decimal}\\
& & & \swarrow\\
& 0.15B & = 60 & \\
& \frac{0.15B}{0.15} & = \frac{60}{0.15} & \leftarrow \text{divide both sides by 0.15}\\
& B & = 400 & \\
\text{Answer:} & \multicolumn{3}{l}{15\% \text{ of } 400 \text{ is } 60}
\end{array}$$

Example 6:

$$\underset{\downarrow}{15}\ \underset{\downarrow}{\text{is}}\ \underbrace{\text{what percent of 60?}}$$

Equation: $A = PB$

$$15 = P(60)$$

$$\frac{15}{60} = \frac{P(60)}{60} \longleftarrow$$ divide both sides by 60

$$0.25 = P$$

$$25\% = P \longleftarrow$$ change 0.25 to percent for final answer

Answer: 15 is 25% of 60

Example 7:

Be careful with *fractional percents*. Instead of converting these percents to decimals, our work will be easier if we convert fractional percents to fractions.

Equation: $16\frac{2}{3}\%$ of $72 = A$

First, change the fractional percent to an improper fraction.

$$16\frac{2}{3}\% = \frac{50}{3} \text{ hundredths or } \frac{\frac{50}{3}}{100} \longleftarrow \text{ means } \frac{50}{3} \div 100$$

$$= \frac{50}{3} \times \frac{1}{100} \longleftarrow$$ multiply by reciprocal

$$= \frac{\overset{1}{\cancel{50}}}{3} \times \frac{1}{\cancel{100}\,2} \longleftarrow$$ cancel by dividing the opposite numerator and denominator (by 50) to write in lowest terms

$$= \frac{1}{6}$$

multiply numerator
multiply denominator

Therefore: $16\frac{2}{3}\%$ of $72 = A$ ⟵ rewrite equation with fraction and multiply

$$\frac{1}{6}(72) = A$$

$$\frac{1}{{}_{1}\cancel{6}} \times \frac{\cancel{72}\,{}^{12}}{1} = A \longleftarrow$$ cancel by dividing the opposite numerator and denominator (by 6) to write in lowest terms

$$\frac{12}{1} = A \longleftarrow$$ multiply numerator
multiply denominator

$$12 = A \longleftarrow$$ reduced and in simplest form

In advertising, we may see signs stating $33\frac{1}{3}\%$ off. Again, we need to convert our percent to a fraction rather than a decimal.

Suppose a \$480 stereo system is reduced $33\frac{1}{3}\%$. We want to know how much money we would save by taking advantage of the discount.

$$33\frac{1}{3}\% =$$

change percent to a fraction

$$\frac{33\frac{1}{3}}{100} =$$

$$33\frac{1}{3} \div 100 =$$

change mixed number to an improper fraction and multiply by reciprocal

$$\frac{100}{3} \text{ x } \frac{1}{100} =$$

cancel by dividing the opposite numerator and denominator (by 100)

$$\frac{{}^{1}\cancel{100}}{3} \text{ x } \frac{1}{\cancel{100}_{1}} =$$

multiply numerator
multiply denominator

$$\frac{1}{3}$$

reduced and in simplest form

Therefore: $33\frac{1}{3}\%$ of \$480 = A

rewrite equation with fraction and multiply

$$\frac{1}{3}(480) = A$$

$$\frac{1}{{}_{1}\cancel{3}} \bullet \frac{\cancel{480}^{160}}{1} = A$$

cancel by dividing the opposite numerator and denominator (by 3) to write in lowest terms

$$\frac{160}{1} = A$$

multiply numerator
multiply denominator

$$160 = A$$

reduced and in simplest form

We would save \$160.

Therefore,

$$\begin{aligned} \$480 &= \text{original cost} \\ -\ 160 &= 33\tfrac{1}{3}\% \text{ off original cost} \\ \hline \$320 &= \text{sale cost} \end{aligned}$$

In the last assignment, we saw that fractions like $\frac{1}{2}$, $\frac{1}{4}$, $\frac{3}{4}$, and $\frac{1}{5}$ convert to **terminating decimals** like 0.5, 0.25, 0.75, and 0.2 respectively. *Terminating decimals* contain a *finite* (limited) number of digits. Therefore, fractional percents like $6\frac{1}{2}\%$, $3\frac{1}{4}\%$, and $12\frac{3}{4}\%$ can easily be converted to decimals in our computations.

Examples:

$$\begin{aligned} 6\tfrac{1}{2}\% &= 6.5\% &&= 0.065 \\ 3\tfrac{1}{4}\% &= 3.25\% &&= 0.0325 \\ 12\tfrac{3}{4}\% &= 12.75\% &&= 0.1275 \\ 18\tfrac{1}{5}\% &= 18.2\% &&= 0.182 \end{aligned}$$

However: $16\frac{2}{3}\% = \frac{1}{6}$ (as explained on page 218)

$33\frac{1}{3}\% = \frac{1}{3}$

Likewise: $66\frac{2}{3}\% = \frac{2}{3}$ because

$66\frac{2}{3}\% =$ ← change percent to a fraction

$\frac{66\frac{2}{3}}{100} =$ ← change mixed number to an improper fraction and multiply

$66\frac{2}{3} \div 100 =$

$\frac{200}{3} \times \frac{1}{100} =$ ← cancel by dividing the opposite numerator and denominator (by 100) to write in lowest terms

$\frac{\overset{2}{\cancel{200}}}{3} \times \frac{1}{\underset{1}{\cancel{100}}} =$ ← multiply numerator
← multiply denominator

$\frac{2}{3}$ ← reduced and in simplest form

You may find it helpful to memorize the following chart of equivalent percents, decimals, and fractions. (See practice on pages 213 and 214 where you computed these equivalents.)

Equivalent Percents, Decimals, and Fractions			
$20\% = 0.2 = \frac{1}{5}$ $40\% = 0.4 = \frac{2}{5}$ $60\% = 0.6 = \frac{3}{5}$ $80\% = 0.8 = \frac{4}{5}$	$25\% = 0.25 = \frac{1}{4}$ $50\% = 0.5 = \frac{1}{2}$ $75\% = 0.75 = \frac{3}{4}$	$12\frac{1}{2}\% = 0.125 = \frac{1}{8}$ $37\frac{1}{2}\% = 0.375 = \frac{3}{8}$ $62\frac{1}{2}\% = 0.625 = \frac{5}{8}$ $87\frac{1}{2}\% = 0.875 = \frac{7}{8}$	$16\frac{2}{3}\% = 0.1\overline{6} = \frac{1}{6}$ $33\frac{1}{3}\% = 0.\overline{3} = \frac{1}{3}$ $66\frac{2}{3}\% = 0.\overline{6} = \frac{2}{3}$ $83\frac{1}{3}\% = 0.8\overline{3} = \frac{5}{6}$
$100\% = 1$			

Most calculators have a percent key—$\boxed{\%}$. Depending upon your calculator, you may be able to use one of the following two key sequences to calculate the percent of a number.

Example:

64% of 75
64 $\boxed{\%}$ $\boxed{\text{x}}$ 75 $\boxed{=}$
or
75 $\boxed{\text{x}}$ 64 $\boxed{\%}$

The result should be 48.

Using What We Know to Examine Percent Equations

Sometimes it can help to examine mathematical steps using an equation to which we already know the answer. For example, if we know that

100% of 48 is 48,

we also know that half or

50% of 48 is 24.

Let's use the statement 50% of 48 equals 24 and reexamine the *fundamental rule for percentage*:

$$\begin{aligned} \text{percent of base} &= \text{amount} \\ PB &= A \end{aligned}$$

and look at solving for *amount*, *base*, and then for *percent*.

First, solving for the *amount* (A)—which we already know from above is **24**:

- What number is 50% of 48?

$$\begin{aligned} A &= P \times B \\ A &= 0.50 \times 48 \\ A &= \mathbf{24} \end{aligned}$$

Answer: **24** is 50% of 48

Second, solving for the *base* (B)—which we already know from above is **48**:

- 50% of what number is 24?

$$\begin{aligned} P \times B &= A \\ 0.50B &= 24 \\ \frac{0.50B}{0.50} &= \frac{24}{0.50} \\ B &= \mathbf{48} \end{aligned}$$

Answer: 50% of **48** is 24

Last, solving for the *percent* (P)—which we already know from the previous page is **50%**.

- 24 is what percent of 48?

$$\begin{aligned} A &= PB \\ 24 &= P(48) \\ \frac{24}{48} &= \frac{(P)(48)}{48} \\ 0.50 &= P \\ \mathbf{50\%} &= P \end{aligned}$$

Answer: 24 is **50%** of 48

If you come across an equation you are unsure of how to solve, go back to an equation you do know the answers to and *plug* those numbers in to help you retrace the steps to the solution.

Practice

Set up **equations** *and* **solve**. *Show all your work.*

1. 25% of 8 is what number?

2. 30 is what percent of 20?

3. 12% of what number is 0.6?

4. 30 is 5% of what number?

5. 27 is what percent of 50?

6. 125% of 36 is what number?

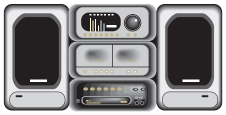
7. 40 is $33\frac{1}{3}$% of what number?

8. $12\frac{1}{2}$% of what number is 6?

9. $16\frac{2}{3}$% of 60 is what number?

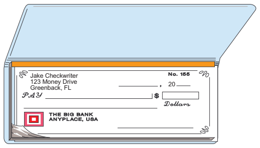
10. 5 is what percent of 40?

11. A stereo that costs $336 requires a down payment of $84. What percent of the total cost is the down payment?



Answer: ______________

12. Jake's $360 paycheck had a $16\frac{2}{3}\%$ deduction for federal income tax. How much money was withheld for income tax?



Answer: ____________

13. $66\frac{2}{3}\%$ of the seats on a bus are filled. There are 45 seats on the bus. How many seats are filled?

Answer: ____________

14. There were 60 questions on the test and Sam answered 90% of them correctly. How many questions did he answer correctly?

Answer: ____________

15. Which of the following is *not* equal to 9?

a. $\frac{3}{4}$ of 12
b. 75% of 16
c. 50% of 18
d. (0.15) x (60)

Answer: ____________

Discount and Sales Tax

Most of us shop enough to know that stores frequently will have sales where they *discount* or mark down items. The discount is the part that you do *not* pay.

Most percent problems can be worked using the *fundamental rule for percentage problems* (*PBA*).

$$\begin{array}{ccccc} \text{percent of base} & = & \text{amount} \\ \downarrow\ \ \downarrow\ \ \downarrow & \downarrow & \downarrow \\ \text{percent} \times \text{base} & = & \text{amount} \\ & \textit{or} & \\ P \times B & = & A \end{array}$$

Example:

A local shoe store offers a 15% discount on shoes if you bring a newspaper coupon. How much would the discount be on a $75 pair of running shoes?

In this case,
the *percent* (or rate) is 15% or 0.15
the *base* (total) is $75, the cost of the shoes.
the *amount* you *don't* pay is the discount.

$$P \times B = A$$
$$(0.15) \times (\$75) = A$$

$11.25 is the *discount* or amount you do *not* pay.

If the shoes once cost $75, and you don't pay $11.25, then the *sales price* will be

cost – discount = sales price

$75 – $11.25 = $63.75

Taxes

It has been said that there are only two things certain in life—death and taxes. In some counties in Florida the residents pay a 7% sales tax. (This rate varies around the state.)

Example:

Let's go back to the running shoes in the last example. To figure the tax, we will use the same *fundamental rule for percentage problems* (*PBA*).

percent x base = amount

In this case,

the *percent* is 7 % or 0.07
the *base* is $63.75, the cost of the shoes
the *amount* is the tax that will be paid to the state of Florida.

P x *B* = *A*

(0.07) x ($63.75) = amount (tax)
$4.4625 = ← tax (round to the nearest cent)
$4.46 = ↙ tax (to be paid)

Not only do we have to pay the store for the shoes ($63.75), but we also have to pay tax ($4.46). Our *total bill* will be

cost + tax = total bill

$63.75 + $4.46 = $68.21

What if our county's sales tax was 6.5% or 0.065? How much would we have to pay on $63.75?

P x *B* = *A*

(0.065) x ($63.75) = amount (tax)
$4.14375 = ← tax (round to the nearest cent)
$4.14 = ↙ tax (to be paid)

cost + tax = total bill

$63.75 + $4.14 = $67.89

Practice

Find the **discount** *on the following. Show all your work.*

1. A $49 dress at 20% off has a *discount* of __________ .

2. A $250 television at 25% off has a *discount* of __________ .

3. A $20 CD at 10% off has a *discount* of __________ .

Find the **sales tax** *on the following if the* **tax rate is 7%**. **Round to the nearest cent.** *Show all your work.*

4. On a $49.50 sweater, the *tax* will be __________ .

5. On an $8,500 car, the *tax* will be __________ .

6. On a $1,500 big-screen television, the *tax* will be __________ .

Find the **total cost** *on the following.* **Round to the nearest cent.** *Show all your work.*

7. The regular price of a pair of jeans is $42. The jeans are now discounted 15%, and the tax rate is 6%. What is the *total cost* of the jeans?

   Answer: __________

8. Find the *total cost* of a $75 jacket which has been discounted 20%. The tax rate is 7.5%.

   Answer: __________

Practice

Answer the following.

Suppose you buy a $10 item for $8 on sale. What is the *rate* or *percent of discount*?

First, $10 – $8 = $2—this is the discount.

Now, use the *fundamental rule of percentages* (*PBA*).

In this case, *P* is *unknown*
B is the *original* or *total price* of the item
A is the *discount*

$P \times B = A$

$P \times \$10 = \$2$ ← rewrite and use commutative property

$10P = 2$ ← divide both sides by 10

$\frac{10P}{10} = \frac{2}{10}$ ←

$P = 0.2$ ← change decimal to percent for the final answer

$P = 20\%$ ←

Notice: Since we are looking for a percent, we need to change 0.2 to 20 %

Find the **discount rate** *for the following.* **Round to the nearest tenth of a percent.** *Show all your work.*

1. A $15 necklace on sale for $10 has a *discount rate* of __________ .

2. A $50 dress on sale for $42 has a *discount rate* of __________ .

Solve the following. **Round to the nearest cent.** *Show all your work.*

3. A store advertises that everything is discounted 15%. You find that a $20 shirt is now $17. Is the store marking items correctly? Explain.

Answer: __________

Explanation: ____

____

____

4. A customer bought a pen for $7.95. The sales tax was 5%. The customer gave the clerk a $10 bill. Will this be enough money to pay for the pen? Why or why not? Explain.

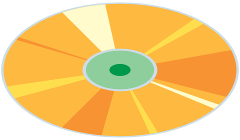
Answer: __________

Explanation: ____

____

____

5. At a used CD store, all CDs are sold at a reduced price of $\frac{1}{2}$ off the new price. This week, the store will take an additional 20% off the *reduced price*. How much would a CD cost that sold new at $15? Explain.



Answer: __________

Explanation: ____

____

____

Interest

Borrowing Money

Banks and individuals lend money to people who qualify because they feel the money will most likely be repaid. The person who borrows money must pay a fee. **Interest** is the fee paid for the use of borrowed money. Usually the *interest* is some percent or rate of the borrowed money. The borrowed money is called the **principal**. The longer a person keeps the money, the bigger fee or greater interest he will pay.

To *compute interest*, we use the following formula:

$$\boxed{I = p \text{ x } r \text{ x } t}$$

p is the *principal*; r is the *rate* (percent); t is the *time* in years

interest = principal • rate • time

Remember: Change the *rate* from a *percent* to a *decimal*.

Example:

Suppose that you need $6,000 for a used car. Your mom agrees to give it to you if you pay her back in 1 year. She also charges you 5% (0.05) annually (per year). How much will the car cost you?

$$\begin{aligned} I &= \quad p \quad\; \text{x} \quad r \quad \text{x} \;\; t \\ I &= (\$6{,}000) \text{ x } (0.05) \text{ x } (1) \\ I &= \$300 \end{aligned}$$

Remember: You still must repay your mom the original $6,000 plus the fee (interest) of $300:

$6,000 + $300 = $6,300
$6,300 is the total amount repaid to your mom.

Money Saving

Also, when you deposit money into a savings account, you are actually agreeing to let the bank use your money. For this privilege, the bank pays *you* interest on your money. To figure this interest, we use the same formula as before:

interest (I) = principal (p) • annual rate of interest (r) • time in years (t)

$$\boxed{\begin{array}{c} I = p \times r \times t \\ \text{or} \\ I = prt \end{array}}$$

Example:

Suppose your mom decides to put her $6,300 in a savings account for 1 year. The bank that she uses pays a rate of $5\frac{3}{4}\%$ (0.0575) on the principal, the amount in the savings account.

$$\begin{array}{lllll} I = & p & \times & r & \times\ t \\ I = & (\$6{,}300) & \times & (0.0575) & \times\ (1) \\ I = & \$362.25 & & & \end{array}$$

When you deposit money into a savings account, you are actually agreeing to let the bank use your money.

At the end of the year your mom still has the original $6,300 plus the $362.25 that she earned in interest.

$6,300 + $362.25 = $6,662.25
principal + interest = *new* principal

Example:

Mr. Powell deposited \$1,000 in his savings account. His account earns $6\frac{1}{2}\%$ (0.065) annually. Unfortunately, Mr. Powell has to withdraw his money after 6 months. How much interest will he earn?

To work this problem we have to deal with the time very carefully. If we use 6 for our *t* in the formula, we will get a large number which is not correct. This would be the answer for allowing the money to stay in the bank 6 years! To get the correct answer, we have to realize that 6 months is $\frac{1}{2}$ year.

$$\frac{\text{6 months}}{\text{12 months in one year}} = \frac{1}{2}$$

$$I = p \times r \times t$$

$$I = (\$1{,}000) \times (0.065) \times \left(\frac{1}{2}\right)$$

$$I = 32.50$$

\$32.50 is the interest that Mr. Powell will earn.

Practice

Solve the following. Show all your work.

1. Joe puts \$3,500 in a savings account which pays $8\frac{1}{2}\%$ annually. Find the amount of interest after 1 year. How much will Joe have in the bank at the end of the year?

2. Sue borrowed \$5,000 from her dad at 6% annually. She repaid him after 9 months. What was the total amount that she repaid?

3. Find the interest on \$50,000 at $5\frac{1}{4}\%$ annually for 6 months.

4. Laura deposits $100 in a savings account that pays 6% annual interest. The chart below is partially completed. Study the chart and figure out what Laura would have at the end of 3, 4, and 5 years. Then complete the chart. **Round to the nearest cent.**

	p **x** ***r*** **x** ***t*** **=** ***I***
Year 1	$100 x 0.06 x 1 = $6, then add $100 + 6 = $106
Year 2	$106 x 0.06 x 1 = $6.36, then add $106 + 6.36 = $112.36
Year 3	$112.36 x 0.06 x 1 = ______ , then add ____ + ____ = ______
Year 4	______ x 0.06 x 1 = ______ , then add ____ + ____ = ______
Year 5	______ x 0.06 x 1 = ______ , then add ____ + ____ = ______

Percent of Increase or Decrease

In 2001, United States businesses sold 9.4 billion dollars worth of video games. This was up from 6.9 billion dollars worth sold in 1999. The new amount is more than the original amount.

Find the **percent of change**—in this case, the **percent of increase**. This is a *percent of increase* because the new amount is more than the original amount.

To find the *percent of change*—*percent of increase* or *decrease*—follow these steps:

Step 1: Subtract to find the *amount of change* or difference between the two amounts.

Step 2: Write the amount of change over the original amount as a fraction and divide.

$$\frac{\text{amount of change}}{\text{original amount}}$$

Remember: The *denominator* in the formula above is *always* the *original amount*, whether less than or greater than the new amount.

Step 3: Change the answer to a percent.

Example 1:

Find out the *percent of increase* to solve the problem above.

Step 1: $9.4 - 6.9 = 2.5$ ⟵ subtract to find the amount of change

Step 2: $\frac{2.5}{6.9} \approx$ ⟵ write as a fraction: $\frac{\text{amount of change (increase)}}{\text{original amount}}$

$0.36 =$ ⟵ divide and round to the nearest hundred

Step 3: 36% ⟵ rewrite the answer as a percent

Answer: 36% increase

Here is another example: Janet weighed 120 pounds when she entered 9th grade. At the end of the year she weighed 105 pounds. The new amount is less than the original amount.

Find the *percent of change*—in this case, the **percent of decrease**. This is a *percent of decrease* because the new amount is less than the original amount. In business, the percent of decrease is often called the *discount*.

Example 2:

Find the *percent of decrease* to solve the problem above.

Step 1: 120 – 105 = 15 ← subtract to find the amount of change

Step 2: $\frac{15}{120}$ = ← write as a fraction: $\frac{\text{amount of change (decrease)}}{\text{original amount}}$

0.125 = ← divide and round to the nearest hundredth

Step 3: 12.5% ← rewrite the answer as a percent

Answer: 12.5% decrease

Practice

Find the **percent of change**. *Then write* **increase** *if it is a* **percent of increase** ***or*** *write* **decrease** *if it is a* **percent of decrease**. **Round to the nearest tenth of a percent** *if necessary. Show all your work.*

1. original amount: \$62; new amount: \$68.50

   Answer: ____________ % of ____________

2. original amount: \$75; new amount: \$50.25

   Answer: ____________ % of ____________

3. original amount: \$900; new amount: \$1200

   Answer: ____________ % of ____________

4. When James bought his new car it was worth \$22,000. Five years later its value was only \$12,500. Find the percent of decrease.

   Answer: ____________ % of decrease

5. Alvin bought a house in 1974 for $39,000. In 2001 he sold it for $98,000. Find the percent of increase.

   Answer: ____________ % of increase

6. A $440 boat is on sale for $390. Find the percent of decrease.

   Answer: ____________ % of decrease

7. In 1980, the population of Denton was about 43,200. In 1990 the population was about 40,100. What was the percent of decrease?

   Answer: ____________ % of decrease

8. In the past 3 decades the average new home has increased in size from 1,500 to 2,300 square feet. Find the percent of increase.

   Answer: ____________ % of increase

9. Leon County had a population of about 100,000 in 1970. In 2000, the county had grown to around 220,000. Find the percent of increase.

   Answer: ____________ % of increase

Circle the letter of the correct answer.

10. Suppose you own 100 shares of stock. The value of each share is $12. After one year, the stock drops (falls) to $9 per share. What is the percent of decrease in the share price of the stock?

    a. 25%

    b. 30%

    c. 12%

    d. $33\frac{1}{3}\%$

Practice

Use the list below to complete the following statements.

amount
denominator
discount
interest
numerator
original
percent (%)
percent of decrease
percent of increase
principal
terminating decimals

1. To change a *fraction to a decimal*, we divide the denominator into the __________________________ .

2. To change a *decimal to a fraction*, simply count the decimal places and use the same number of zeros in the __________________________ .

3. A __________________________ is a fraction whose denominator is 100.

4. The *fundamental rule for percentage problems* (*PBA*) is percent x base = __________________________ .

5. Fractions like $\frac{1}{2}$, $\frac{1}{4}$, $\frac{3}{4}$, and $\frac{1}{5}$ convert to __________________________ with a finite (limited) number of digits.

6. __________________________ is the fee paid for the use of borrowed money.

7. Usually the interest is some percent or rate of the borrowed money and the borrowed money is called the __________________________ .

8. The percent the amount of *decrease* is of the original amount is called the ____________________ or the ____________________ .

9. The percent the amount of *increase* is of the original amount is called the ____________________ .

10. The percent of change is the amount of change divided by the ____________________ amount.

Practice

Match each definition with the correct term. Write the letter on the line provided.

______ 1. a fraction that has a numerator greater than or equal to the denominator

______ 2. any number representing some part of a whole; of the form $\frac{a}{b}$

______ 3. a fraction whose numerator and denominator have no common factor greater than 1

______ 4. a mathematical sentence stating that two ratios are equal

______ 5. any number written with a decimal point in the number

______ 6. a shorthand method of writing very large or very small numbers using exponents in which a number is expressed as the product of a power of 10 and a number that is greater than or equal to one (1) and less than 10

______ 7. a number that consists of both a whole number and a fraction

______ 8. the quotient of two numbers used to compare two quantities

______ 9. a mathematical sentence that equates one expression to another expression

A. decimal number

B. equation

C. fraction

D. improper fraction

E. mixed number

F. proportion

G. ratio

H. scientific notation

I. simplest form

Unit Review

Part A

Answer the following.

1. Solve: $\frac{7}{8} = \frac{?}{24}$

2. Write in simplest form: $\frac{24}{32}$

3. Write in simplest form: $\frac{7x^2y}{14xy}$

4. Change the whole number 4 to sixths.

5. Change $3\frac{2}{5}$ to an improper fraction.

6. Change $\frac{16}{3}$ to a mixed number.

7. Change $\frac{7}{8}$ to a decimal.

8. Change 0.008 to a fraction in simplest form.

Fill in the blank with >, <, *or* =.

9. $\frac{5}{4}$ _______ $\frac{17}{12}$

10. $\frac{1}{2}$ _______ 0.54

Add ***or*** **subtract** *as indicated. Write each answer in* **simplest form***.*

11. $\frac{8}{9} + \frac{2}{9} =$

12. $6\frac{1}{4} + 2\frac{3}{4} =$

13. $\frac{3}{8} + \frac{4}{5} =$

14. $3\frac{1}{4} - 1\frac{1}{2} =$

15. $8 + 3.6 + 2.45 =$

16. $18 - 6.2 =$

Multiply ***or*** **divide** *as indicated. Write each answer in* **simplest form***.*

17. $\frac{1}{9} \bullet \frac{3}{4} =$

18. $(3\frac{1}{2})(2\frac{2}{5}) =$

19. $\frac{5}{6} \div \frac{5}{3} =$

20. $6\frac{1}{2} \div 2 =$

21. $6.3 \text{ x } 0.21 =$

22. $2.75 \div 0.5 =$

Write in **scientific notation***.*

23. One ounce is equal to 0.000031104 metric tons. ________________

24. A millennium has 31,536,000,000 seconds. ____________________

Write each **ratio** *as a* **fraction** *in* **simplest form***.*

25. 14:20 ____

26. 3 inches to 15 inches ____

Unit Review

Part B

Answer the following. Check for reasonableness of results. Show all your work.

27. James and Joel drove 200 miles in 5 hours. What was their rate (or average speed)?

    Answer: ____________ miles per hour

28. Rachel reached her goal of running 4 miles in 40 minutes. What was Rachel's rate (or average speed)? **Hint:** 1 hour = 60 minutes

    Answer: ____________ miles per hour

29. Find the unit price of a 12-ounce package of chips which costs 78 cents. **Round to the nearest cent.**

    Answer: ____________ per ounce

30. Solve this proportion: $\frac{8}{\frac{1}{4}} = \frac{24}{n}$

    Answer: ____________

31. Set up a proportion and solve: Marlon spent \$18 for 12 gallons of gas. At that price, how many gallons could he buy for \$22.50?

Answer: ____________ gallons

Fill in the following chart.

	Fraction	**Decimal**	**Percent**
32.	$\frac{4}{5}$		
33.		0.016	
34.			120%
35.			$6\frac{1}{2}\%$
36.		0.08	
37.	$\frac{5}{4}$		

Set up an equation using the **fundamental rule of percentages (PBA)**.

percent of base is amount
or
PB = A

Then **solve** *each equation. Check for reasonableness of results. Show all your work.*

38. What number is 15% of 120?

39. 15% of what number is 120?

40. 15 is what percent of 120?

41. Find the discount on an $80 dress which has been discounted 15%. Then determine the sale price of the dress.

amount of discount: ______________________

sales price: ______________________

42. Find the sales tax on a television costing $160.50, if the tax rate is 7%. Then determine the total cost of the television including the tax. **Round to the nearest cent.**

sales tax: ____________________

total cost including tax: ____________________

Use the **formula to figure interest** *for the following problem. Show all your work.*

Interest (I) = principal (p) x rate (r) x time (t)

$I = p \times r \times t$

43. Emily deposited $10,000 in a bank that pays $5\frac{1}{4}\%$ annual interest. She kept the money there for 9 months. How much interest was earned? What was the total amount that Emily received from the bank?

interest earned: ____________________

total amount: ____________________

Use $\frac{\text{amount of change}}{\text{original amount}}$ *for the following problems. Check for reasonableness of results.*

44. Carolyn bought a house 10 years ago and paid $80,000. She recently sold it for $100,000. Find the *percent of increase*.

    Show all your work.

    Answer: ____________

45. Jerry bought a new Buick for $20,000 in 1995. In 2001, his daughter wrecked the car and the insurance company "totaled it out." He received a check in the mail for $7,000. Find the *percent of decrease*.

    Show all your work.

    Answer: ____________

Unit 3: Algebraic Thinking

This unit emphasizes strategies used to solve equations and understand and solve inequalities.

Unit Focus

Number Sense, Concepts, and Operations

- Understand the relative size of integers, rational numbers, irrational numbers, and real numbers. (MA.A.1.4.2)
- Understand that numbers can be represented in a variety of equivalent forms, including integers, fractions, decimals, percents, scientific notation, exponents, radicals, and absolute value. (MA.A.1.4.4)
- Understand and explain the effects of addition, subtraction, multiplication, and division on real numbers, including square roots, exponents, and appropriate inverse relationships. (MA.A.3.4.1)
- Select and justify alternative strategies, such as using properties of numbers, including inverse, identity, distributive, associative, and transitive, that allow operational shortcuts for computational procedures in real-world or mathematical problems. (MA.A.3.4.2)
- Add, subtract, multiply, and divide real numbers, including square roots and exponents, using appropriate methods of computing, such as mental mathematics, paper and pencil, and calculator. (MA.A.3.4.3)

Measurement

- Use concrete and graphic models to derive formulas for finding perimeter, area, surface area, circumference, and volume of two-dimensional shapes. (MA.B.1.4.1)

Algebraic Thinking

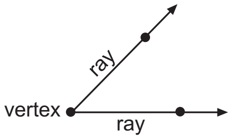
- Describe, analyze, and generalize relationships, patterns, and functions using words, symbols, variables, tables, and graphs. (MA.D.1.4.1)
- Use systems of equations and inequalities to solve real-world problems algebraically. (MA.D.2.4.2)

Data Analysis and Probability

- Interpret data that has been collected, organized, and displayed in tables. (MA.E.1.4.1)

Vocabulary

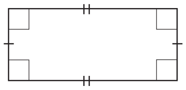
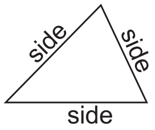
Use the vocabulary words and definitions below as a reference for this unit.

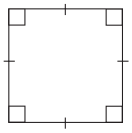
additive identity	the number zero (0), that is, adding 0 does not change a number's value *Example*: 5 + 0 = 5
additive inverses	a number and its opposite whose sum is zero (0); also called *opposites* *Example*: In the equation 3 + -3 = 0, 3 and -3 are additive inverses, or *opposites*, of each other.
angle (∠)	the shape made by two rays extending from a common endpoint, the vertex; measures of angles are described in degrees (°) 
area (*A*)	the inside region of a two-dimensional figure measured in square units *Example:* A rectangle with sides of four units by six units contains 24 square units or has an area of 24 square units.
associative property	the way in which three or more numbers are grouped for addition or multiplication does *not* change their sum or product *Example*: (5 + 6) + 9 = 5 + (6 + 9) *or* (2 x 3) x 8 = 2 x (3 x 8)
commutative property	the order in which any two numbers are added or multiplied does *not* change their sum or product *Example*: 2 + 3 = 3 + 2 *or* 4 x 7 = 7 x 4

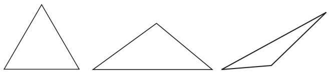
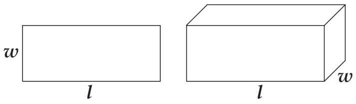
consecutive	in order *Example*: 6, 7, 8 are consecutive whole numbers and 4, 6, 8 are consecutive even numbers.
cube (power)	the third power of a number *Example*: $4^3 = 4 \text{ x } 4 \text{ x } 4 = 64$
cubic units	units for measuring volume
decrease	to make less
degree (º)	common unit used in measuring angles
difference	the result of a subtraction *Example*: In $16 - 9 = 7$, 7 is the difference.
distributive property	for any real numbers a, b, and x, $x(a + b) = ax + bx$
equation	a mathematical sentence that equates one expression to another expression *Example*: $2x = 10$
equivalent (forms of a number)	the same number expressed in different forms *Example*: $\frac{3}{4}$, 0.75, and 75%
even number	any whole number divisible by 2 *Example*: 2, 4, 6, 8, 10, 12 …

expression	a collection of numbers, symbols, and/or operation signs that stands for a number *Example*: $4r^2$; $3x + 2y$; Expressions do *not* contain equality (=) or inequality (<, >, ≤, ≥, or ≠) symbols.
graph of a number	the point on a number line paired with the number
increase	to make greater
inequality	a sentence that states one expression is greater than (>), greater than or equal to (≥), less than (<), less than or equal to (≤), or not equal to (≠) another expression *Example*: $a \neq 5$ or $x < 7$
integers	the numbers in the set {… , -4, -3, -2, -1, 0, 1, 2, 3, 4, …}
inverse operation	an action that cancels a previously applied action *Example*: Subtraction is the inverse operation of addition.
irrational number	a real number that cannot be expressed as a ratio of two numbers *Example*: $\sqrt{2}$
length (*l*)	a one-dimensional measure that is the measurable property of line segments
like terms	terms that have the same variables and the same corresponding exponents *Example*: In $5x^2 + 3x^2 + 6$, $5x^2$ and $3x^2$ are like terms

measure (m) of an angle (∠)	the number of degrees (°) of an angle
multiplicative identity	the number one (1), that is, multiplying by 1 does not change a number's value *Example*: $5 \times 1 = 5$
multiplicative inverses	any two numbers with a product of 1; also called *reciprocals* *Example*: 4 and $\frac{1}{4}$
multiplicative property of -1	the product of any number and -1 is the opposite or additive inverse of the number *Example*: $-1(a) = -a$ and $a(-1) = -a$
multiplicative property of zero	for any number a, $a \bullet 0 = 0$ and $0 \bullet a = 0$
negative numbers	numbers less than zero
number line	a line on which numbers can be written or visualized
odd number	any whole number *not* divisible by 2 *Example*: 1, 3, 5, 7, 9, 11 …
order of operations	the order of performing computations in parentheses first, then exponents or powers, followed by multiplication and division, then addition and subtraction *Example*: $5 + (12 - 2) \div 2 - 3 \times 2 =$ $5 + 10 \div 2 - 3 \times 2 =$ $5 + 5 - 6 =$ $10 - 6 =$ 4

perimeter (*P*)	the length of the boundary around a figure; the distance around a polygon
positive numbers	numbers greater than zero
power (of a number)	an exponent; the number that tells how many times a number is used as a factor *Example*: In 2^3, 3 is the power.
product	the result of a multiplication *Example*: In 6 x 8 = 48, 48 is the product.
quotient	the result of a division *Example*: In 42 ÷ 7 = 6, 6 is the quotient.
ratio	the quotient of two numbers used to compare two quantities *Example*: The ratio of 3 to 4 is $\frac{3}{4}$.
rational number	a real number that can be expressed as a ratio of two integers
real numbers	all rational and irrational numbers
reciprocals	two numbers whose product is 1 *Example*: Since $\frac{3}{4} \times \frac{4}{3} = 1$, the reciprocal of $\frac{3}{4}$ is $\frac{4}{3}$.
rectangle	a parallelogram with four right angles 
side	the edge of a two-dimensional geometric figure *Example*: A triangle has three sides. 

simplify an expression	to perform as many of the indicated operations as possible
solution	any value for a variable that makes an equation or inequality a true statement *Example*: In $y = 8 + 9$ $y = 17$ 17 is the solution.
solve	to find all numbers that make an equation or inequality true
square	a rectangle with four sides the same length 
square (of a number)	the result when a number is multiplied by itself or used as a factor twice *Example*: 25 is the square of 5.
square units	units for measuring area; the measure of the amount of an area that covers a surface
substitute	to replace a variable with a numeral *Example*: $8(a) + 3$ $8(5) + 3$
substitution property of equality	for any numbers a and b, if $a = b$, then a may be replaced by b
sum	the result of an addition *Example*: In $6 + 8 = 14$, 14 is the sum.

symmetric property of equality	for any numbers a and b, if $a = b$, then $b = a$
table (or chart)	an orderly display of numerical information in rows and columns
triangle	a polygon with three sides 
variable	any symbol that could represent a number
width (w)	a one-dimensional measure of something side to side 

Unit 3: Algebraic Thinking

Introduction

Algebraic thinking provides tools for looking at situations. You can state, simplify, and show relationships through algebraic thinking. Using algebraic thinking and algebraic symbols, you can record ideas and gain insights into situations.

In this lesson you will use what you have learned about solving equations to solve equations involving positive and negative numbers.

You have learned to solve equations such as these:

$y + 12 = 36$ $\qquad$ $y - 12 = 36$ $\qquad$ $12y = 36$ $\qquad$ $\frac{y}{12} = 36$

Lesson One Purpose

- Understand and explain the effects of addition, subtraction, multiplication, and division on real numbers, including square roots, exponents, and appropriate inverse relationships. (MA.A.3.4.1)
- Add, subtract, multiply, and divide real numbers, including square roots and exponents, using appropriate methods of computing, such as mental mathematics, paper and pencil, and calculator. (MA.A.3.4.3)
- Use concrete and graphic models to derive formulas for finding perimeter, area, surface area, circumference, and volume of two-dimensional shapes. (MA.B.1.4.1)
- Use systems of equations and inequalities to solve real-world problems algebraically. (MA.D.2.4.2)

Solving Equations

A mathematical sentence that contains an equal sign (=) is called an **equation**. In Unit 1 and Unit 2, we learned that an *equation* equates one **expression** to another expression.

We also learned the rules to add and subtract and to multiply and divide **positive numbers** and **negative numbers**.

Rules to Add and Subtract Positive and Negative Integers

(+)	+	(+)	=	+	(+)	-	(+)	=	positive if first number is greater, otherwise it is negative
(–)	+	(–)	=	–	(–)	-	(–)	=	negative if first number is greater, otherwise it is positive
(+)	+	(–)	=	use sign of **integer** with greatest absolute value	(+)	-	(–)	=	+
(–)	+	(+)	=		(–)	-	(+)	=	–

Rules to Multiply and Divide Positive and Negative Integers

(+)	•	(+)	=	+	(+)	÷	(+)	=	+
(–)	•	(–)	=	+	(–)	÷	(–)	=	+
(+)	•	(–)	=	–	(+)	÷	(–)	=	–
(–)	•	(+)	=	–	(–)	÷	(+)	=	–

To **solve** the equation is to find the number that we can **substitute** for the **variable** to make the equation true.

Study these examples. Each equation has been *solved* and then checked by substituting the answer for the variable in the original equation. If the answer makes the equation a true sentence, it is called the **solution** of the equation.

Solve:

$$\begin{aligned} n + 14 &= -2 \\ n + 14 - 14 &= -2 - 14 \\ n &= -2 + -14 \\ n &= -16 \end{aligned}$$

Check:

$$\begin{aligned} n + 14 &= -2 \\ -16 + 14 &= -2 \\ -2 &= -2 \quad \textit{It checks!} \end{aligned}$$

Solve:

$$\begin{aligned} y - (-6) &= 2 \\ y + 6 - 6 &= 2 - 6 \\ y &= 2 + -6 \\ y &= -4 \end{aligned}$$

Check:

$$\begin{aligned} y - -6 &= 2 \\ -4 - -6 &= 2 \\ -4 + 6 &= 2 \\ 2 &= 2 \quad \textit{It checks!} \end{aligned}$$

Solve:

$$\begin{aligned} -6x &= -66 \\ \frac{-6x}{-6} &= \frac{-66}{-6} \\ x &= 11 \end{aligned}$$

Check:

$$\begin{aligned} -6x &= -66 \\ -6(11) &= -66 \\ -66 &= -66 \quad \textit{It checks!} \end{aligned}$$

Solve:

$$\begin{aligned} \frac{y}{-10} &= 5 \\ (-10)\frac{y}{-10} &= 5(-10) \\ y &= -50 \end{aligned}$$

Check:

$$\begin{aligned} \frac{y}{-10} &= 5 \\ \frac{-50}{-10} &= 5 \\ 5 &= 5 \quad \textit{It checks!} \end{aligned}$$

Practice

Solve *each equation and* **check**. *Show* **essential steps**.

1. $y + 12 = 2$

2. $a - -2 = 2$

3. $r + 15 = -25$

4. $0 = y + -46$

5. $15y = -30$

6. $\frac{y}{15} = -2$

7. $\frac{x}{5} = -9$

8. $-9y = 270$

9. $m - 9 = -8$

10. $3 + x = -3$

11. $\frac{n}{-5} = -2$

12. $-55 = -5a$

13. $12 = -6 + x$

14. $t - 20 = -15$

Interpreting Words and Phrases

Words and phrases can suggest relationships between numbers and mathematical operations. In Unit 1 and Unit 2, we learned how words and phrases can be translated into mathematical expressions. (See pages 38-39 and page 215.) Appendix B also contains a list of mathematical symbols and their meanings.

Relationships between numbers can be indicated by words such as **consecutive**, *preceding*, *before*, and *next*. Also, the same mathematical expression can be used to translate many different word expressions.

Below are some of the words and phrases we associate with the four mathematical operations and with powers of a number.

Mathematical Symbols and Words

+	–	x	÷	power
add sum plus total more than increased by	subtract difference minus remainder less than decreased by	multiply product times of twice doubled	divide quotient	power square cube

Practice

Write an **equation** *and* **solve** *the problem.*

*Exampl*e: Sixteen less than a number *n* is 48. What is the number?

Remember: The word *is* means *is equal to* and translates to an = sign.

$$\begin{aligned}
\underbrace{\text{16 less than a number } n} &\;=\; 48 \\
&\downarrow \quad \downarrow \\
n - 16 &= 48 \\
n - 16 + 16 &= 48 + 16 \\
n &= 64
\end{aligned}$$

Note: To write 16 less than *n*, you write *n* – 16.

So 64 – 16 = 48 *or* 16 less than 64 is 48.

1. A number increased by 9 equals -7. What is the number? (Let *d* = the number).

2. A number times -12 equals -72. What is the number? (Let *x* = the number.)

3. A number decreased by 5 equals -9. What is the number? (Let *y* = the number.)

4. A number divided by 7 equals -25. What is the number? (Let n = the number.)

5. In a card game, Ann made 30 points on her first hand. After the second hand, her total score was 20 points. What was her score on the second hand?

6. A scuba diver is at the -30 foot level. How many feet will she have to rise to be at the -20 foot level?

Solving Two-Step Equations

When solving an equation, you want to get the *variable* by itself on one side of the equal sign. You do this by *undoing* all the operations that were done on the variable. In general, undo the addition or subtraction first. Then undo the multiplication or division.

Study the following examples.

A. Solve:

$$\begin{aligned} 2y + 2 &= 30 \\ 2y + 2 - 2 &= 30 - 2 && \text{subtract 2 from each side} \\ \frac{2y}{2} &= \frac{28}{2} && \text{divide each side by 2} \\ y &= 14 \end{aligned}$$

Check:

$$\begin{aligned} 2y + 2 &= 30 \\ 2(14) + 2 &= 30 && \text{replace } y \text{ with 14} \\ 28 + 2 &= 30 \\ 30 &= 30 && \textit{It checks!} \end{aligned}$$

B. Solve:

$$\begin{aligned} 2x - 7 &= -29 \\ 2x - 7 + 7 &= -29 + 7 && \text{add 7 to each side} \\ \frac{2x}{2} &= \frac{-22}{2} && \text{divide each side by 2} \\ x &= -11 \end{aligned}$$

Check:

$$\begin{aligned} 2x - 7 &= -29 \\ 2(-11) - 7 &= -29 && \text{replace } x \text{ with -11} \\ -22 - 7 &= -29 \\ -29 &= -29 && \textit{It checks!} \end{aligned}$$

C. Solve:

$\frac{n}{7} + 18 = 20$

$\frac{n}{7} + 18 - 18 = 20 - 18$ subtract 18 from each side

$\cancel{(7)}\frac{n}{\cancel{7}} = 2(7)$ multiply each side by 7 and

$n = 14$ cancel the 7s on the left

Check:

$\frac{n}{7} + 18 = 20$

$\frac{14}{7} + 18 = 20$ replace n with 14

$2 + 18 = 20$

$20 = 20$ *It checks!*

D. Solve:

$\frac{t}{-2} + 4 = -10$

$\frac{t}{-2} + 4 - 4 = -10 - 4$ subtract 4 from each side

$\frac{\cancel{(-2)}\, t}{\cancel{-2}} = -14(-2)$ multiply each side by -2 and

$t = 28$ cancel the -2s on the left

Check:

$\frac{t}{-2} + 4 = -10$

$\frac{28}{-2} + 4 = -10$ replace t with 28

$-14 + 4 = -10$

$-10 = -10$ *It checks!*

Practice

Solve *each equation and* **check**. *Show* **essential steps**.

1. $4x + 8 = 16$

2. $4y - 6 = 10$

3. $5n + 3 = -17$

4. $2y - 6 = -18$

5. $-8y - 21 = 75$

6. $\frac{a}{8} - 17 = 13$

7. $13 + \frac{x}{-3} = -4$

8. $\frac{n}{8} + 1 = 4$

9. $-3b + 5 = 20$

10. $6 = \frac{x}{4} - 14$

11. $-7y + 9 = -47$

12. $\frac{n}{-6} - 17 = -8$

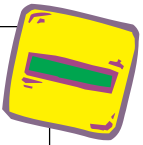
Use the list below to decide which **equation** *to use to solve each problem. Then* **solve** *the problem.*

Equation A: $\frac{n}{4} + 2 = 10$

Equation B: $\frac{n}{4} - 2 = 10$

Equation C: $4n + 2 = 10$

Equation D: $4n - 2 = 10$



13. Two more than the product of 4 and Ann's age is 10.

    Equation: ______________________

    How old is Ann? ___________

14. If you multiply Sean's age by 4 and then subtract 2, you get 10.

    Equation: ______________________

    What is Sean's age? ___________

15. If you divide Joe's age by 4 and then add 2, you get 10.

Equation: ____________________________

What is Joe's age? ______________

16. Divide Jenny's age by 4, then subtract 2, and you get 10.

Equation: ____________________________

What is Jenny's age? ______________

Circle the letter of the correct answer.

17. The sentence that means the same as the equation $\frac{1}{3}y + 8 = 45$ is ______________ .

a. Eight *more than* one-third of y is 45.
b. One-third of y is eight *more than* 45.
c. y is eight *less than* one-third of 45.
d. y is eight *more than* one-third of 45.

Special Cases

Reciprocals: Two Numbers Whose Product is 1

Note: $5 \bullet \frac{1}{5} = 1$ and $\frac{5}{5} = 1$

Multiplying 5 by $\frac{1}{5}$ and dividing 5 by 5, both yield 1.

We see that 5 is the **reciprocal** of $\frac{1}{5}$ and $\frac{1}{5}$ is the *reciprocal* of 5. Every number but zero has a reciprocal. (Division by zero is undefined.) Two numbers are reciprocals if their product is 1.

Here are some examples of numbers and their reciprocals:

Number	Reciprocal
$-\frac{1}{4}$	-4
1	1
$\frac{-2}{3}$	$\frac{-3}{2}$
$\frac{7}{8}$	$\frac{8}{7}$
-2	$-\frac{1}{2}$
$\frac{1}{7}$	7
x	$\frac{1}{x}$

Multiplication Property of Reciprocals

any nonzero number times its reciprocal is 1

$$x \bullet \frac{1}{x} = 1$$

If $x \neq 0$

Remember: When two numbers are reciprocals of each other, they are also called **multiplicative inverses** of each other.

Study the following two examples:

Method 1: Division Method

$$\begin{aligned} 5x - 6 &= 9 \\ 5x - 6 + 6 &= 9 + 6 \\ 5x &= 15 \\ \tfrac{5x}{5} &= \tfrac{15}{5} \\ x &= 3 \end{aligned}$$

Method 2: Reciprocal Method

$$\begin{aligned} 5x - 6 &= 9 \\ 5x - 6 + 6 &= 9 + 6 \\ 5x &= 15 \\ \tfrac{1}{5} \bullet 5x &= \tfrac{1}{5} \bullet 15 \\ x &= 3 \end{aligned}$$

Both methods work well. However, the *reciprocal method* is probably easier in the next two examples, which have fractions:

$$\begin{aligned} -\tfrac{1}{5}x - 1 &= 9 \\ -\tfrac{1}{5}x - 1 + 1 &= 9 + 1 \\ -\tfrac{1}{5}x &= 10 \\ \mathbf{-5} \bullet -\tfrac{1}{5}x &= \mathbf{-5} \bullet 10 && \text{multiply by reciprocal of } -\tfrac{1}{5} \text{ which is -5} \\ x &= -50 \end{aligned}$$

Here is another equation with fractions:

$$\begin{aligned} -\tfrac{3}{4}x + 12 &= 36 \\ -\tfrac{3}{4}x + 12 - 12 &= 36 - 12 \\ -\tfrac{3}{4}x &= 24 \\ -\tfrac{\mathbf{4}}{\mathbf{3}} \bullet -\tfrac{3}{4}x &= -\tfrac{\mathbf{4}}{\mathbf{3}} \bullet 24 && \text{multiply by reciprocal of } -\tfrac{3}{4} \text{ which is } -\tfrac{4}{3} \\ 1 \bullet x &= -32 \\ x &= -32 \end{aligned}$$

Multiplying by -1

Here is another equation which sometimes gives people trouble:

$5 - x = -10$

Remember: $5 - x$ is not the same thing as $x - 5$. To work this equation we need to make the following observation:

Property of Multiplying by -1

-1 times a number equals the opposite of that number

$$-1 \bullet x = -x$$

This property is also called the **multiplicative property of -1** which says the *product* of any number and -1 is the opposite or **additive inverse** of the number.

For example:

$-1 \bullet 5 = -5$
$-1 \bullet (-6) = 6$

Now let's go back to $5 - x = -10$ using the property of multiplying by -1. We can rewrite the equation as

$$\begin{aligned} 5 - 1x &= -10 \\ 5 - 1x \mathbf{- 5} &= -10 \mathbf{- 5} && \text{subtract 5 from both sides to} \\ -1x &= -15 && \text{isolate the variable} \\ \frac{-1x}{-1} &= \frac{-15}{-1} \\ x &= 15 \end{aligned}$$

This example requires great care with the positive numbers and negative signs:

$$\begin{aligned} 11 - \tfrac{1}{9}x &= -45 \\ 11 - \tfrac{1}{9}x \mathbf{- 11} &= -45 \mathbf{- 11} && \text{subtract 11 from both sides} \\ -\tfrac{1}{9}x &= -56 && \text{to isolate the variable} \\ \mathbf{-9} \bullet -\tfrac{1}{9}x &= \mathbf{-9} \bullet -56 && \text{multiply by reciprocal of } -\tfrac{1}{9} \text{ which is } -9 \\ x &= 504 \end{aligned}$$

Consider the following example:

Remember: *Decreased by* means *subtract*, *product* means *multiply*, and *is* translates to the = sign.

Five decreased by the product of 7 and x is -6. Solve for x.

$$\text{Five } \underbrace{\text{decreased by}}_{-} \text{ the } \underbrace{\text{product of 7 and } x}_{7x} \underbrace{\text{ is }}_{=} -6.$$

$$\begin{aligned} 5 \quad - \quad 7x &= -6 \\ 5 - 7x \mathbf{- 5} &= -6 \mathbf{- 5} && \text{subtract 5 from both sides to} \\ -7x &= -11 && \text{isolate the variable} \\ \frac{-7x}{-7} &= \frac{-11}{-7} \\ x &= \frac{11}{7} \text{ or } 1\tfrac{4}{7} \end{aligned}$$

Practice:

Write the **reciprocals** *of the following.*

1. 10
2. -6
3. $\frac{5}{6}$
4. $-\frac{9}{10}$
5. 0

Solve *the following. Show* **essential steps**.

6. $\frac{1}{5}x + 3 = 9$

9. $10 - 6x = 11$

7. $\frac{1}{4}x - 7 = 2$

10. $15 - x = 10$

8. $-\frac{1}{2}x - 7 = 23$

11. $\frac{1}{8}x + 4 = -6$

12. $-6 - x = 10$

14. $4 - \frac{3}{7}x = 10$

13. $2 + \frac{5}{6}x = -8$

Check yourself: *Use the list of* **scrambled answers** *below and check your answers to problems 6-14.*

-80 -60 -16 -14 -12 $-\frac{1}{6}$ **5 30 36**

Answer the following.

15. The difference between 12 and $2x$ is -8. Solve for x.

Practice

Match each definition with the correct term. Write the letter on the line provided.

______ 1. to find all numbers that make an equation or inequality true

______ 2. numbers less than zero

______ 3. a collection of numbers, symbols, and/or operation signs that stands for a number

______ 4. any value for a variable that makes an equation or inequality a true statement

______ 5. any symbol that could represent a number

______ 6. to replace a variable with a numeral

______ 7. a mathematical sentence that equates one expression to another expression

______ 8. numbers greater than zero

______ 9. two numbers whose product is 1

A. equation

B. expression

C. negative numbers

D. positive numbers

E. reciprocals

F. solution

G. solve

H. substitute

I. variable

Practice

Use the list below to write the correct term for each definition on the line provided.

additive inverses	**multiplicative inverses**
decrease	**multiplicative property of -1**
difference	**product**
increase	

______________________ 1. any two numbers with a product of 1

______________________ 2. the result of a multiplication

______________________ 3. to make greater

______________________ 4. to make less

______________________ 5. the product of any number and -1 is the opposite or additive inverse of the number

______________________ 6. the result of a subtraction

______________________ 7. a number and its opposite whose sum is zero (0); also called *opposites*

Lesson Two Purpose

- Understand and explain the effects of addition, subtraction, multiplication, and division on real numbers, including square roots, exponents, and appropriate inverse relationships. (MA.A.3.4.1)
- Select and justify alternative strategies, such as using properties of numbers, including inverse, identity, distributive, associative, and transitive, that allow operational shortcuts for computational procedures in real-world or mathematical problems. (MA.A.3.4.2)
- Add, subtract, multiply, and divide real numbers, including square roots and exponents, using appropriate methods of computing, such as mental mathematics, paper and pencil, and calculator. (MA.A.3.4.3)
- Use concrete and graphic models to derive formulas for finding perimeter, area, surface area, circumference, and volume of two-dimensional shapes. (MA.B.1.4.1)
- Use systems of equations and inequalities to solve real-world problems algebraically. (MA.D.2.4.2)

The Distributive Property

Consider 4(2 + 6). The rules for **order of operations** would have us add inside parentheses first.

$$\begin{array}{cc} 4(2 + 6) & = \\ 4(8) & = \\ 32 & \end{array}$$

Remember: Order of operations—

1. parentheses
2. powers (exponents)
3. multiplication *or* division
4. addition *or* subtraction

However, there is a second way to do the problem.

$$\begin{aligned} 4(2+6) &= \\ 4(2)+4(6) &= \\ 8 \;+\; 24 &= \\ 32& \end{aligned}$$

In this way, the 4 is being *distributed* over the addition. The second way of doing the problem illustrates the **distributive property**.

The Distributive Property

For any numbers a, b, and c,
$a(b + c) = ab + ac$

Also, it works for subtraction:
$a(b - c) = ab - ac$

This property is most useful in simplifying expressions that contain variables, such as $2(x + 4)$.

To **simplify an expression** we must perform as many of the indicated operations as possible. However, in the expression $2(x + 4) =$, we can't add first, unless we know what number x represents. The *distributive property* allows us to rewrite the equation:

$$\begin{aligned} 2(x+4) &= \\ 2x + 2(4) &= \\ 2x + 8& \end{aligned}$$

The distributive property allows you to multiply each term *inside* a set of parentheses by a factor *outside* the parentheses. Multiplication is *distributive over* addition and subtraction.

$$\begin{aligned} 5(3+1) &= (5 \bullet 3) + (5 \bullet 1) \\ 5(4) &= 15 + 5 \\ 20 &= 20 \end{aligned} \qquad \begin{aligned} 5(3-1) &= (5 \bullet 3) - (5 \bullet 1) \\ 5(2) &= 15 - 5 \\ 10 &= 10 \end{aligned}$$

Not all operations are distributive. You cannot distribute division over addition.

$$\begin{aligned} 14 - (5 + 2) &\neq 14 \div 5 + 14 \div 2 \\ 14 \div 7 &\neq 2.8 + 7 \\ 2 &\neq 9.8 \end{aligned}$$

In Unit 1 we learned about other properties that help us work with variables. See page 51 and study the chart below.

Properties

Addition		**Multiplication**	
Commutative:	$a + b = b + a$	Commutative:	$ab = ba$
Associative:	$(a + b) + c = a + (b + c)$	Associative:	$(ab)c = a(bc)$
Identity:	0 is the identity. $a + 0 = a$ and $0 + a = a$	Identity:	1 is the identity. $a \bullet 1 = a$ and $1 \bullet a = a$
Addition		**Subtraction**	
Distributive:	$a(b + c) = ab + ac$ and $(b + c)a = ba + ca$	Distributive:	$a(b - c) = ab - ac$ and $(b - c)a = ba - ca$

These properties deal with the following:

order—**commutative property** of addition and commutative property of multiplication

grouping—**associative property** of addition and associative property of multiplication

identity—**additive identity** property and **multiplicative identity** property

zero—**multiplicative property of zero**

distributive—**distributive property** of multiplication over addition and over subtraction

Notice in the distributive property that it does *not* matter whether $\boldsymbol{a}$ is placed on the *right* or the *left* of the expression in parentheses.

$$\boldsymbol{a}(b + c) = (b + c)\boldsymbol{a} \text{ or } \boldsymbol{a}(b - c) = (b - c)\boldsymbol{a}$$

The **symmetric property of equality** (if $a = b$, then $b = a$) says that if one quantity equals a second quantity, then the second quantity also equals the first quantity. We use the **substitution property of equality** when replacing a variable with a number or when two quantities are equal and one quantity can be replaced by the other. Study the chart and examples below describing properties of equality.

Properties of Equality

Reflexive: $a = a$
Symmetric: If $a = b$, then $b = a$.
Transitive: If $a = b$ and $b = c$, then $a = c$.
Substitution: If $a = b$, then a may be replaced by b.

Examples of Properties of Equality

Reflexive: $8 – e = 8 – e$
Symmetric: If $5 + 2 = 7$, then $7 = 5 + 2$.
Transitive: If $9 – 2 = 4 + 3$ and $4 + 3 = 7$, then $9 – 2 = 7$.
Substitution: If $x = 8$, then $x \div 4 = 8 \div 4$. x is replaced by 8.

or

If $9 + 3 = 12$, then $9 + 3$ may be replaced by 12.

Consider this expression to simplify.

$5(6x + 3) + 8 =$	use the distributive property to
$5(6x) + 5(3) + 8 =$	distribute 5 over $6x$ and 3
$30x + 15 + 8 =$	use the associative property to
$30x + 23$	associate 15 and 8

and

$6 + 2(4x – 3) =$	use order of operations to multiply before adding, then
$6 + 2(4x) + 2(-3) =$	distribute 2 over $4x$ and -3
$6 + 8x + -6 =$	use the associative property to associate 6 and -6
$8x + 0 =$	use the identity property of addition
$8x$	

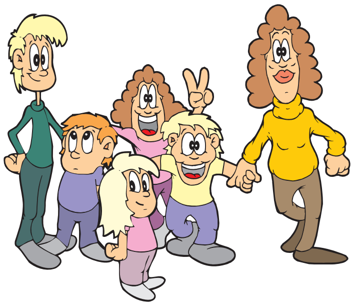
Practice

Simplify *by using the* **distributive property**. *Show* **essential steps**.

1. $10(x + 9)$
2. $16(z - 3)$
3. $a(b + 5)$
4. $5(x + 3) + 9$
5. $4(x - 5) + 20$
6. $5(3 + x) - 9$
7. $4(3x + 7) - 2$
8. $-6(x + 3) + 18$
9. $30 + 2(x + 8)$
10. $x(x + 3)$
11. $a(b + 10)$

Circle the letter of the correct answer.

12. Mrs. Smith has 5 children. Each fall she buys each child a new book bag for $20, a new notebook for $3.50, and other school supplies for $15. Which expression is a correct representation for the amount she spent?



    a. 5(20 + 3.50 + 15)

    b. 5 + (20 + 3.50 + 15)

    c. 5(20 • 3.50 • 15)

    d. 5 • 20 + 3.50 + 15

Number the **order of operations** *in the correct* **order**. *Write the numbers 1-4 on the line provided.*

______ 13. addition *or* subtraction

______ 14. powers (exponents)

______ 15. parentheses

______ 16. multiplication *or* division

Simplifying Expressions

Here's how to use the distributive property and the definition of subtraction to simplify the following expressions.

Example 1:

Simplify

$-7a - 3a$

$$\begin{aligned} -7a - 3a &= -7a + -3a \\ &= (-7 + -3)a && \text{use the distributive property} \\ &= -10a \end{aligned}$$

Example 2:

Simplify

$10c - c$

$$\begin{aligned} 10c - c &= 10c - 1c \\ &= 10c + -1c \\ &= (10 + -1)c && \text{use the distributive property} \\ &= 9c \end{aligned}$$

The expressions $-7a - 3a$ and $-10a$ are called **equivalent** *expressions*. The expressions $10c - c$ and $9c$ are also called *equivalent* expressions. Equivalent expressions express the same number. An expression is in simplest form when it is replaced by an equivalent expression having no **like terms** and no parentheses.

Study these examples.

$$\begin{aligned} -5x + 4x &= (-5 + 4)x \\ &= -x \end{aligned}$$

$$\begin{aligned} 5y - 5y &= 5y + -5y \\ &= (5 + -5)y \\ &= 0y && \text{multiplicative property of zero} \\ &= 0 \end{aligned}$$

The multiplicative property of 0 says for any number *a*,

$$a \bullet 0 = 0 \bullet a = 0.$$

Frequently, the following shortcut is used to simplify expressions.

First

- change each subtraction to adding the opposite
- then combine *like terms* (terms that have the same variable) by adding.

Simplify

$2a + 3 - 6a$

like terms

$$\begin{aligned} 2a + 3 - 6a &= 2a + 3 + -6a \\ &= -4a + 3 \end{aligned}$$

change – 6*a* to + -6*a*

combine like terms by adding

Simplify

$8b + 7 - b - 6$

like terms

$$\begin{aligned} 8b + 7 - b - 6 &= 8b + 7 + -1b + -6 \\ &= 7b + 1 \end{aligned}$$

like terms

change – *b* to + -1*b* and – 6 to + -6

combine like terms by adding

Simplify

$7x + 5 + 3x = 10x + 5$ combine like terms

like terms

Practice

Simplify *by combining* **like terms**. *Show* **essential steps**.

1. $5n + 3n$

2. $6n - n$

3. $8y - 8y$

4. $7n + 3n - 6$

5. $-7n - 3n - 6$

6. $6n - 3 + 7$

7. $4x + 11x$

8. $4x - 11x$

9. $-4x - 11x$

10. $10y - 4y + 7$

11. $10y + 4y - 7$

12. $10y - 4 - 7$

13. $8c - 12 - 6c$

14. $8c - 12c - 6$

15. $-10y - y - 15$

16. $-10y + y - 15$

17. $15x - 15x + 6$

18. $15x - 15 + 8x$

19. $20n - 6 - n + 8$

20. $20n - 6n - 1 + 8$

21. $12c - 15 - 12c - 17$

22. $12c - 15c - 12 - 18$

Practice

Match each definition with the correct term. Write the letter on the line provided.

______ 1. $a(b + c) = ab + ac$

______ 2. $a + b = b + a$

______ 3. $(a + b) + c = a + (b + c)$

A. associative property

B. commutative property

C. distributive property

______ 4. $a \bullet 1 = a$

______ 5. $a + 0 = a$

______ 6. $a \bullet 0 = 0$

A. additive identity

B. multiplicative identity

C. multiplicative property of zero

______ 7. if $a = b$, then $b = a$

______ 8. if $a = b$, then a may be replaced by b

A. substitution property of equality

B. symmetric property of equality

______ 9. terms that have the same variables and the same corresponding exponents

______ 10. to perform as many of the indicated expressions as possible

______ 11. the order of performing computations in parentheses first, then exponents or powers, followed by multiplication and division, then addition and subtraction

A. like terms

B. order of operations

C. simplify an expression

Equations with Like Terms

Consider the following equation.

$$2x + 3x + 4 = 19$$

Look at both sides of the equation and see if either side can be simplified.

> Always simplify first
> by combining like terms.

$$\begin{aligned}
2x + 3x + 4 &= 19 \\
5x + 4 &= 19 && \text{add like terms} \\
5x + 4 - 4 &= 19 - 4 && \text{subtract 4 from each side} \\
5x &= 15 \\
\tfrac{5x}{5} &= \tfrac{15}{5} && \text{divide each side by 5} \\
x &= 3
\end{aligned}$$

Always mentally check your answer by *substituting* the solution for the variable in the original equation.

Substitute 3 for x in the equation.

$$\begin{aligned}
2x + 3x + 4 &= 19 \\
2(3) + 3(3) + 4 &= 19 \\
6 + 9 + 4 &= 19 \\
19 &= 19 && \textit{It checks!}
\end{aligned}$$

Consider this example:

The product of x and 7 plus the product of x and 3 is 45.

Remember: To work a problem like this one, we need to remember two things. The word *product* means *multiply* and the word *is* always translates to =.

$$\text{The }\underbrace{\text{product of }x\text{ and }7}\text{ plus the }\underbrace{\text{product of }x\text{ and }3}\text{ is }45.$$

$$\begin{aligned} 7x + 3x &= 45 & \\ 10x &= 45 & \text{add like terms} \\ \frac{10x}{10} &= \frac{45}{10} & \text{divide both sides by 10} \\ x &= 4.5 & \end{aligned}$$

Check by substituting 4.5 for x in the original equation.

$$\begin{aligned} 7x + 3x &= 45 & \\ 7(4.5) + 3(4.5) &= 45 & \\ 31.5 + 13.5 &= 45 & \\ 45 &= 45 & \textit{It checks!} \end{aligned}$$

Here is another example which appears to be a little harder:

$$\begin{aligned} 3x - 2 - x + 10 &= -12 & \\ 3x - 2 - 1x + 10 &= -12 & \text{remember: } 1 \bullet x = x \\ 3x - 1x - 2 + 10 &= -12 & \text{add like terms} \\ 2x + 8 &= -12 & \\ 2x + 8 - 8 &= -12 - 8 & \text{subtract 8 from both sides} \\ 2x &= -20 & \\ \frac{2x}{2} &= \frac{-20}{2} & \text{divide both sides by 2} \\ x &= -10 & \end{aligned}$$

Check by substituting -10 into the original equation.

$$\begin{aligned} 3x - 2 - x + 10 &= -12 & \\ 3(-10) - 2 - (-10) + 10 &= -12 & \\ -30 - 2 + 10 + 10 &= -12 & \\ -32 + 20 &= -12 & \\ -12 &= -12 & \textit{It checks!} \end{aligned}$$

Practice

Solve *these equations by first* **simplifying** *each side. Show* **essential steps**.

1. $4x + 6x = -30$

2. $-2x + 10x - 6x = -12$

3. $12m - 6m + 4 = -32$

4. $3 = 4x + x - 2$

5. $3y - y - 8 = 30$

6. $x + 10 - 2x = -2$

7. $13x + 105 - 8x = 0$

8. $2x + 10 + 3x - 8 = -13$

✓ **Check yourself:** Add all your answers for problems 1-8. Did you get a *sum of* $-4\frac{1}{3}$? If yes, complete the practice. If no, correct your work before continuing.

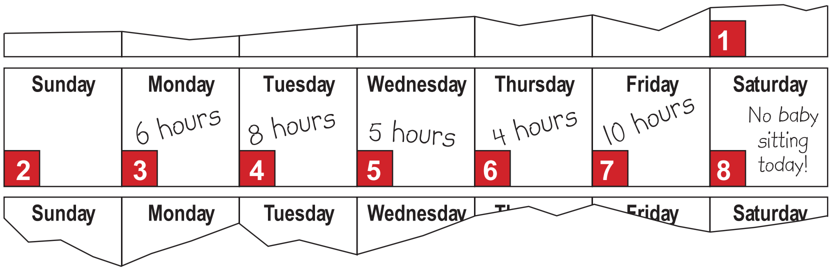
Write an **equation** *to represent each* **situation**. *Then* **solve** *the equation* **for** x. *Show* **essential steps**.

9. The sum of $2x$, $3x$, and $5x$ is 50.

10. The difference of $3x$ and $8x$ is -15.

11. The sum of $6x$, $-2x$, and $10x$, decreased by 15 is 13.

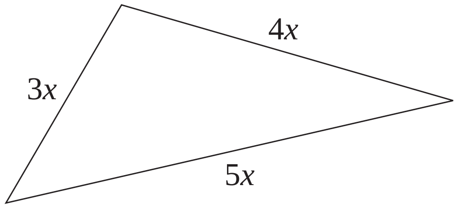
12. Your neighbor hired you to baby-sit for $\$x$ per hour. Here is a record of last week:



Your total salary for the week was \$198.00. How much do you earn per hour?

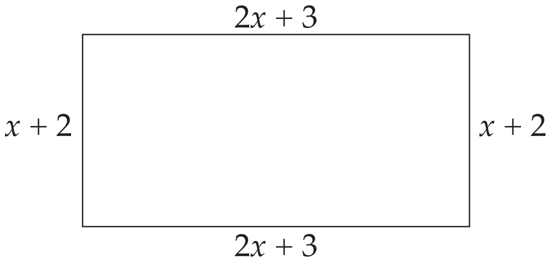
13. The **perimeter (*P*)** of the **triangle** is 48 inches. What is x?

Remember: The *perimeter* of a figure is the **length (*l*)** of the boundary around a figure or the *sum* of the *lengths* of the **sides**.



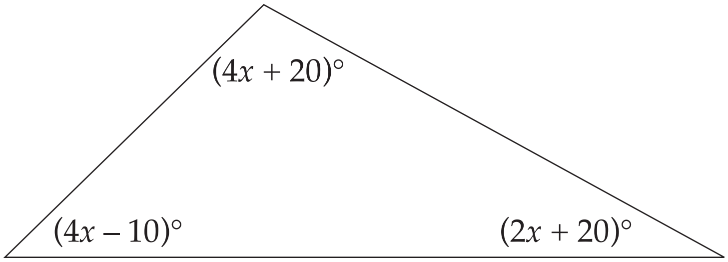
14. Use the answer from problem 13 to find the length of each side of the triangle. Do the sides add up to 48 inches?

15. The perimeter of the **rectangle** is 58 inches. What is x?



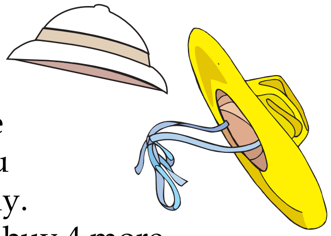
16. Use the answer from problem 15 to find the length and **width (*w*)** of the rectangle. Do the sides add up to 58 inches?

17. In any triangle, the sum of the **measures (m) of the angles (∠)** is always 180 **degrees (°)**. What is x?



18. Using the answer from problem 17, find the measure of each **angle**. Do the angles add up to 180 degrees?

Circle the letter of the correct answer.



19. You and your friend go to a popular theme park in Central Florida. The admission for the two of you comes to a total of $70. Both of you immediately buy 2 hats to wear during the day. Later, as you are about to leave you decide to buy 4 more hats for your younger brothers and sisters who didn't get to come. The total bill for the day is $115.00. Which equation could you use to find the cost of a single hat?

   a. $6x = 115$
   b. $70 + 2x + 4x = 115$
   c. $70 – 6x = 115$
   d. $70 + 2x = 115 + 6x$

Complete the following.

20. A common mistake in algebra is to say that

    $3x + 4x = 7x^2$, instead of

    $3x + 4x = 7x$.

    Let $x = 2$ and substitute into both expressions below.

Remember: When you are doing $7x^2$ the rules for the order of operation require that you square *before* you multiply!

$$x = 2$$

$3x + 4x = 7x$

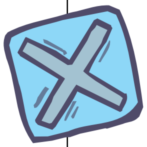
$3x + 4x = 7x^2$

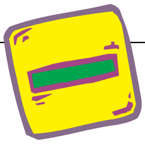
Are you convinced that $3x + 4x$ is not equal to ($\neq$) $7x^2$?

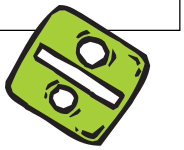
Putting It All Together

Guidelines for Solving Equations

1. Use the distributive property to clear parentheses.
2. Combine like terms. We want to isolate the variable.
3. Undo addition or subtraction using **inverse operations**.
4. Undo multiplication or division using *inverse operations*.
5. Check by substituting the solution in the original equation.







SAM = **S**implify (steps 1 and 2) then
Add (or subtract)
Multiply (or divide)

Example 1

Solve:

$$\begin{aligned}
6y + 4(y + 2) &= 88 \\
6y + 4y + 8 &= 88 && \text{use distributive property} \\
10y + 8 - 8 &= 88 - 8 && \text{combine like terms and undo addition} \\
& && \text{by subtracting 8 from each side} \\
\frac{10y}{10} &= \frac{80}{10} && \text{undo multiplication by dividing by 10} \\
y &= 8
\end{aligned}$$

Check solution in the original equation:

$$\begin{aligned}
6y + 4(y + 2) &= 88 \\
6(8) + 4(8 + 2) &= 88 \\
48 + 4(10) &= 88 \\
48 + 40 &= 88 \\
88 &= 88 && \textit{It checks!}
\end{aligned}$$

Example 2

Solve:

$$\begin{aligned}
-\tfrac{1}{2}(x+8) &= 10 \\
-\tfrac{1}{2}x-4 &= 10 && \text{use distributive property} \\
-\tfrac{1}{2}x-4+4 &= 10+4 && \text{undo subtraction by adding 4 to} \\
& && \text{both sides} \\
-\tfrac{1}{2}x &= 14 \\
(-2)-\tfrac{1}{2}x &= 14(-2) && \text{isolate the variable by multiplying} \\
x &= -28 && \text{each side by the reciprocal of } -\tfrac{1}{2}
\end{aligned}$$

Check solution in the original equation:

$$\begin{aligned}
-\tfrac{1}{2}(x+8) &= 10 \\
-\tfrac{1}{2}(-28+8) &= 10 \\
-\tfrac{1}{2}(-20) &= 10 \\
10 &= 10 && \textit{It checks!}
\end{aligned}$$

Example 3

Solve:

$$\begin{aligned}
26 &= \tfrac{2}{3}(9x\ -\ 6) \\
26 &= \tfrac{2}{3}(9x)-\tfrac{2}{3}(6) && \text{use distributive property} \\
26 &= 6x-4 \\
26+4 &= 6x-4+4 && \text{undo subtraction by adding 4 to} \\
& && \text{each side} \\
\tfrac{30}{6} &= \tfrac{6x}{6} && \text{undo multiplication by dividing} \\
& && \text{each side by 6} \\
5 &= x
\end{aligned}$$

Check solution in the original equation:

$$\begin{aligned}
26 &= \tfrac{2}{3}(9x\ -\ 6) \\
26 &= \tfrac{2}{3}(9 \bullet 5-6) \\
26 &= \tfrac{2}{3}(39) \\
26 &= 26 && \textit{It checks!}
\end{aligned}$$

Example 4

Solve:

$$\begin{aligned}
x - (2x + 3) &= 4 && \\
x - 1(2x + 3) &= 4 && \text{use the multiplication property of -1} \\
x - 2x - 3 &= 4 && \text{use the multiplicative identity of 1} \\
 & && \text{and use the distributive property} \\
-1x - 3 &= 4 && \text{combine like terms} \\
-1x - 3 + 3 &= 4 + 3 && \text{undo subtraction} \\
\frac{-1x}{-1} &= \frac{7}{-1} && \text{undo multiplication} \\
x &= -7 &&
\end{aligned}$$

Examine the solution steps above. See the use of the *multiplicative property of -1* in front of the parentheses on line two.

line 1: $x - (2x + 3) = 4$
line 2: $x \mathbf{-1}(2x + 3) = 4$

Also notice the use of *multiplicative identity* on line three.

line 3: $\mathbf{1}x - 2x - 3 = 4$

The simple variable x was multiplied by 1 ($1 \bullet x$) to equal $1x$. The $1x$ helped to clarify the number of variables when combining like terms on line four.

Check solution in the original equation:

$$\begin{aligned}
x - (2x + 3) &= 4 && \\
-7 - (2 \bullet -7 + 3) &= 4 && \\
-7 - (-11) &= 4 && \\
4 &= 4 && \textit{It checks!}
\end{aligned}$$

Practice

Solve *and* **check**. *Show* **essential steps**.

1. $10(2n + 3) = 130$

2. $4(y - 3) = -20$

3. $\frac{x}{-2} + 4 = -10$

4. $6x + 6(x - 4) = 24$

5. $6 = \frac{2}{3}(3n - 6)$

6. $10p - 4(p - 7) = 42$

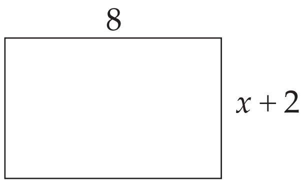
7. $28 = \frac{1}{2}(x - 8)$

8. $x - (3x - 7) = 11$

Solve *and* **Check. Show essential steps.**

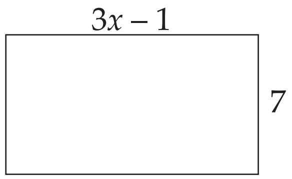
9. Write an equation for the area of the rectangle. Then solve for x.

Remember: To find the **area (A)** of a rectangle, multiply the length (l) times the width (w). $A = (lw)$



Area is 64 **square units**.

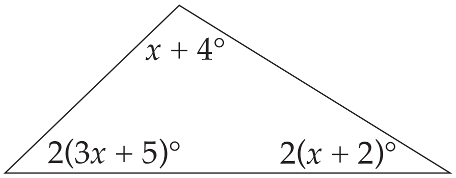
10. Write an equation for the area of the rectangle. Then solve for x.



Area is 56 square units.

11. Write an equation for the measure of degrees (°) in the triangle. Then solve for x.

Remember: For any triangle, the sum of the measures of the angles is 180 degrees.



Practice

Use the list below to write the correct term for each definition on the line provided.

angle
area (*A*)
degree (°)
inverse operation
length (*l*)
measure (m) of an angle(∠)
perimeter (*P*)
rectangle
side
square (of a number)
square units
sum
triangle
width (*w*)

________________________ 1. a one-dimensional measure that is the measurable property of line segments

________________________ 2. a parallelogram with four right angles

________________________ 3. a one-dimensional measure of something side to side

________________________ 4. the result of an addition

________________________ 5. the length of the boundary around a figure

________________________ 6. the shape made by two rays extending from a common endpoint, the vertex

________________________ 7. a polygon with three sides

________________________ 8. the inside region of a two-dimensional figure measured in square units

________________________ 9. the number of degrees (°) of an angle

________________________ 10. units for measuring area; the measure of the amount of an area that covers a surface

______________________ 11. common unit used in measuring angles

______________________ 12. the edge of a two-dimensional geometric figure

______________________ 13. an action that cancels a previously applied action

______________________ 14. the result when a number is multiplied by itself or used as a factor twice

Lesson Three Purpose

- Select and justify alternative strategies, such as using properties of numbers, including inverse, identity, distributive, associative, and transitive, that allow operational shortcuts for computational procedures in real-world or mathematical problems. (MA.A.3.4.2)
- Use systems of equations and inequalities to solve real-world problems algebraically. (MA.D.2.4.2)
- Interpret data that has been collected, organized, and displayed in tables. (MA.E.1.4.1)

Solving Equations with Variables on Both Sides

I am thinking of a number. If you multiply my number by 3 and then subtract 2, you get the same answer that you do when you add 4 to my number. What is my number?

To solve this riddle, begin by translating these words into an algebraic sentence. Let x represent my number.

If you multiply my number by 3 and then subtract 2
$3x - 2$

you get the same answer
$=$

that you do when you add 4 to my number
$4 + x$

Putting it all together, we get the equation $3x - 2 = 4 + x$. Note that this equation is different from equations in previous units. There is a *variable* on both sides. To solve such an equation, we do what we've done in the past. Make sure both sides are simplified, and that there are no parentheses.

Strategy: Collect the variables on one side. Collect the numbers on the other side.

Now let's go back to the equation which goes with our riddle.

Solve:

$3x - 2 = 4 + x$	both sides are simplified
$3x - 2 = 4 + 1x$	multiplicative identity of 1
$3x - 2 - 1x = 4 + 1x - 1x$	collect variables on the left
$2x - 2 = 4$	combine like terms; simplify
$2x - 2 + 2 = 4 + 2$	collect numbers on the right
$\frac{2x}{2} = \frac{6}{2}$	divide both sides by 2
$x = 3$	

Check solution in the original equation and the original riddle:

$3x - 2 = 4 + x$	
$3 \bullet 3 - 2 = 4 + 3$	
$9 - 2 = 7$	
$7 = 7$	*It checks!*

Study the equation below.

Solve:

$2(3x + 4) = 5(x - 2)$	
$6x + 8 = 5x - 10$	distributive property
$6x + 8 - 5x = 5x - 10 - 5x$	variables on the left
$x + 8 = -10$	simplify
$x + 8 - 8 = -10 - 8$	numbers on the right
$x = -18$	

Check solution in the original equation:

$2(3x + 4) = 5(x - 2)$	
$2(3 \bullet -18 + 4) = 5(-18 - 2)$	
$2(-50) = 5(-20)$	
$-100 = -100$	*It checks!*

Let's work this next example in two different ways.

1. Collect the *variables* on the *left* and the *numbers* on the *right*.

Solve:

$$\begin{aligned} 6y &= 4(5y - 7) && \\ 6y &= 20y - 28 && \text{distributive property} \\ 6y - 20y &= 20y - 28 - 20y && \text{variables on the left} \\ \frac{-14y}{-14} &= \frac{-28}{-14} && \text{divide both sides by -14} \\ y &= 2 && \end{aligned}$$

Check solution in the original equation:

$$\begin{aligned} 6y &= 4(5y - 7) && \\ 6 \bullet 2 &= 4(5 \bullet 2 - 7) && \\ 12 &= 4 \bullet 3 && \\ 12 &= 12 && \textit{It checks!} \end{aligned}$$

2. Collect the *variables* on the *right* and *numbers* on the *left*.

Solve:

$$\begin{aligned} 6y &= 4(5y - 7) && \\ 6y &= 20y - 28 && \text{distributive property} \\ 6y - 6y &= 20y - 28 - 6y && \text{variables on the right} \\ 0 &= 14y - 28 && \text{simplify} \\ 0 + 28 &= 14y - 28 + 28 && \text{numbers on the left} \\ \frac{28}{14} &= \frac{14y}{14} && \text{divide both sides by 14} \\ 2 &= y && \end{aligned}$$

We get the *same* answer, so the choice of which side you put the variable on is up to you!

Practice

Solve each **equation** *below. Then* **find** *your* **solution** *at the bottom of the page. Write the* **letter** *next to the number of that equation on the line provided above the solution. Then you will have the answer to this question:*

Which great explorer's last words were,

"I have not told half of what I saw!"

r 1. $2x - 4 = 3x + 6$

l 2. $2(-12 - 6x) = -6x$

a 3. $x - 3 = 2(-11 + x)$

c 4. $-7(1 - 4m) = 13(2m - 3)$

p 5. $-2x + 6 = -x$

m 6. $7x = 3(5x - 8)$

o 7. $2(12 - 8x) = 1x - 11x$

8. ___ ___ ___ ___ ___ ___ ___ ___ ___
3 19 -10 -16 4 6 4 -4 4

✓ **Check yourself:** Use the answer above to check your solutions to problems 1-7. Did your solutions spell out the great explorer's name? If not, correct your work before continuing.

Solve *and* **check**. *Show* **essential steps**.

9. Six more than 5 times a number is the same as 9 less than twice the number. Find the number.

10. Twelve less than a number is the same as 6 decreased by 8 times the number. Find the number.

11. The product of 5 and a number, plus 17, is equal to twice the sum of the same number and -5. Find the number.

Complete *to* **solve** *the following.*

12. $-12 + 7(x + 3) = 4(2x - 1) + 3$

Remember: Always multiply before you add.

-12 + ______ + ______ = ______ – ______ + 3

$7x$ + ______ = $8x$ – ______

Now finish the problem.

13. $-56 + 10(x - 1) = 4(x + 3)$

14. $5(2x + 4) + 3(-2x - 3) = 2x + 3(x + 4)$

______ + ______ + ______ – ______ = $2x$ + ______ + ______
distributive property distributive property

______ $x + 11$ = ______ x + ______
add like terms add like terms

Now finish the problem.

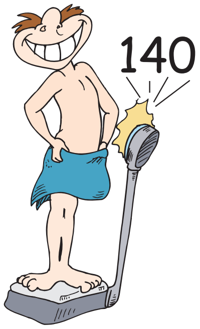
15. $-16x + 10(3x - 2) = -3(2x + 20)$

$-16x +$ ______ – ______ = ______ – ______
distributive property   distributive property

Add like terms and finish the problem.

16. $6(-2x - 4) + 2(3x + 12) = 37 + 5(x - 3)$

Problems That Lead to Equations



Joshua presently weighs 100 pounds, but is on a diet where he gains 2 pounds per week. When will he weigh 140 pounds?

The answer is 20 weeks. Let's use this simple problem to help us think algebraically.

Step 1: Read the problem and label the variable. Underline all clues.

Joshua presently weighs 100 pounds, but is on a diet where he gains 2 pounds per week. When will he weigh 140 pounds?

Let x represent the number of weeks.

Step 2: Plan.

Let $2x$ represent the weight Joshua will gain.

Step 3: Write the equation.

$$\begin{array}{ccccc} \text{present weight} & + & \text{gain} & = & \text{desired weight} \\ 100 & + & 2x & = & 140 \end{array}$$

Step 4: Solve the equation.

$$\begin{aligned} 100 + 2x &= 140 && \\ 100 + 2x - 100 &= 140 - 100 && \text{add -100 to both sides} \\ 2x &= 40 && \\ \tfrac{2x}{2} &= \tfrac{40}{2} && \text{divide both sides by 2} \\ x &= 20 && \end{aligned}$$

Step 5: Check your solution. Does your answer make sense?

$100 \text{ (now)} + 2(20) \text{ weight gain} = 140$

We will use this 5-step approach on the following problems. You will find that many times a picture or chart will also help you arrive at an answer. Remember, we are learning to think algebraically and to do that the procedure is as important as the final answer!

5-Step Plan for Thinking Algebraically

Step 1: **Read** the problem and **label** the variable. **Underline** all clues.

Decide what x represents.

Step 2: **Plan.**

Step 3: **Write** an equation.

Step 4: **Solve** the equation.

Step 5: **Check** your solution. Does your answer make sense?

Practice

Use the **5-step plan** *to solve and check the following. Show* **essential steps***.*

1. Leon's television breaks down, and unfortunately he has only *$100.00* in his savings account for emergencies. The repairman charges *$35.00 for coming to Leon's house* and then another *$20.00 per hour* for fixing the television. *How many hours* can Leon pay for the repairman to work?

   a. Step 1: Read the problem and label the variable. Underline all clues. (*Note*: The clues have been italicized for you.)

      Let x represent ____________________________________ .

   b. Step 2: Plan. Let $20x$ represent ________________________

      ____ .

   c. Step 3: Write an equation. ____________________________

   d. Step 4: Solve the equation.

   e. Step 5: Check your solution. Does it make sense? __________

2. Samantha charges $16.00 to deliver sand to your house, plus $3.50 per **cubic** foot for the sand that you buy. How much sand can you buy for $121.00?

   a. Step 1: Read the problem and label the variable. Underline all clues.

      Let *x* represent ____ .

   b. Step 2: Plan. Let ______ *x* represent ______________________

      ____ .

   c. Step 3: Write an equation. _______________________________

   d. Step 4: Solve the equation.

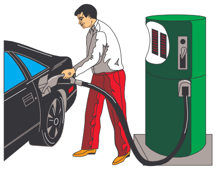
   e. Step 5: Check your solution. Does it make sense? __________

   f. If the installation of a child's sand box requires 29 cubic feet of sand, will you be able to complete this project for $121.00?

      Explain. ____

      ____

3. Suppose that the gas tank of a car holds 20 gallons, and that the car uses $\frac{1}{10}$ of a gallon per mile. How far has the car gone when 5 gallons remain?



a. Step 1: Read the problem and label the variable. Underline all clues.

Let x represent ______________________________________ .

b. Step 2: Plan. Let $\frac{1}{10}x$ represent ___________________________

____ .

c. Step 3: Explain why the appropriate equation is $20 - \frac{1}{10}x = 5$.

____

____

d. Step 4: Solve the equation.

e. Step 5: Check your solution. Does it make sense? __________

4. Jared weighs 250 pounds and is on a diet where he loses 3 pounds per week. How long will it take him to weigh 150 pounds?

   a. Step 1: Read the problem and label the variable. Underline all clues.

      Let x represent ____ .

   b. Step 2: Let $3x$ represent ________________________________ .

   c. Step 3: Write an equation. ______________________________

      ____

   d. Step 4: Solve the equation.

   e. Step 5: Check your solution. Does it make sense? __________

5. Batman has $100.00 and spends $3.00 per day. Robin has $20.00 but is adding to it at the rate of $5.00 per day. When will they have the same amount of money?

   a. Step 1: Read the problem and label the variable. Underline all clues.

      Let x represent ____ .

   b. Step 2: Plan. 100 – 3x is what Batman will have after x days. How much will Robin have after x days? ______________________

      ____

   c. Step 3: Write an equation stating that Batman's money is the same as Robin's money after x days.

      ____

   d. Step 4: Solve the equation.

   e. What does your answer mean? __________________________

      ____

      ____

      ____

   f. Step 5: Check your answer. Does the answer make sense?

      ____

6. Suppose you live in Tallahassee, Florida where the temperature is 84 degrees and going down 3 degrees per hour. A friend lives in Sydney, Australia where the temperature is 69 degrees and going up at a rate of 2 degrees per hour. How long would you and your friend have to wait before the temperatures in both places are equal?

   a. Step 1: Read the problem and label the variable. Underline all clues.

      Let x represent ____ .

   b. Step 2: Plan. Let ________________________ represent

      Tallahassee's temperature and ________________________

      represent Sydney's temperature.

   c. Step 3: Write an equation. Let the Tallahassee temperature *equal*

      Sydney's temperature. __________________________________

   d. Step 4: Solve the equation.

   e. Step 5: Check your solution. Does it make sense? __________

Practice

Use the **5-step plan** *to* **solve** *and* **check** *the following. Show* **essential steps**.

Sometimes a *chart* helps *organize* the information in a problem.

1. I am thinking of *3 numbers*. The *second number is 4 more than the first number*. The *third number is twice the first number*. The *sum* of all 3 numbers *is 28*. Find the numbers.

   Algebraic Thinking:

- Step 1: Read the problem and label the variable. Underline all clues. (*Note*: The clues have been italicized for you.)

  What does x represent? Since the second and third numbers are described in terms of the first number, let x represent the *first* number.

- Step 2: Plan. See the **table** below.

Description		**Value**	
first number	x	=	
second number	$x + 4$	=	
third number	$2x$	=	
sum	$4x + 4$	=	28

- Step 3: Write an equation.

  $4x + 4 = 28$

- Step 4: Solve the equation.

  a. The equation $4x + 4 = 28$ will give you the *value* of only the *first* number. Substitute your answer back into the *expressions* in the *table* on the previous page to find the *second* and *third* numbers.

  b. solve equation to find the value of the *first* number:

     $4x + 4 = 28$

  c. substitute the first number's value in the expression from the *table* on the previous page to get the value of the *second* number:

     $x + 4$

  d. substitute the first number's value in the expression from the *table* on the previous page to get the value of the *third* number:

     $2x$

- Step 5: Check your solution. Does it make sense?

  e. Check solution in original equation.

  f. Do your numbers add up to 28? ______

     Write an equation and solve to prove that the sum of the 3 numbers equal 28.

Sometimes a *picture* helps *organize* the information in a problem.

2. A triangle has a perimeter (P) of 30 inches. The longest side is 8 inches longer than the shortest side. The third side is 1 inch shorter than the longest side. Find the sides.

Remember: The *perimeter* is the sum of all the lengths of all sides.

- Step 1: Read the problem and label the variable. Underline all clues.

  Hint: Let the shortest side be x inches long.

- Step 2: Plan.

  a. Draw a triangle. Label the shortest side x. Label the other two sides in terms of x.

  b. Let ______________________________ represent adding

  up the sides of the triangle.

- Step 3: Write an equation.

  c. Use the fact that the perimeter is 30 inches to write an equation.

  ____

- Step 4: Solve the equation.

  d. Find *x* by solving the equation.

- Step 5: Check your solution. Does it make sense?

  e. Check solution in original equation.

  f. Use the value of *x* to find the other 2 sides.

  g. Do the sides add up to 30? ______

  Write an equation and solve to prove that the sum of the 3 sides of the triangle equals 30.

3. A rectangle has a perimeter of 38 inches and a width of x inches. The length of the rectangle is *4 more than twice the width*. Label all 4 sides.

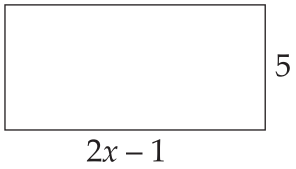
   Draw and label a rectangle and use the 5-step plan to find the dimensions of the rectangle.

4. The measure of the angles in any triangle add up to 180 degrees. Let the smallest angle be x degrees. The second angle is *twice the smallest*. The third angle is *30 degrees more than the second angle*. Find all angles.

   Draw and label a triangle. Use the 5-step plan to find all angles.

5. Write an equation for the area of the rectangle. ________________

Remember: To find the *area* of a rectangle we multiply the length times the width. $A = l \bullet w$

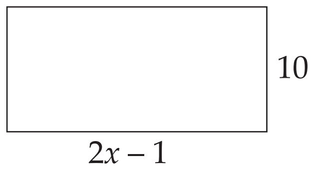


The area is 35 square inches.

Solve the equation for x, then substitute it in $2x - 1$ to find the length.

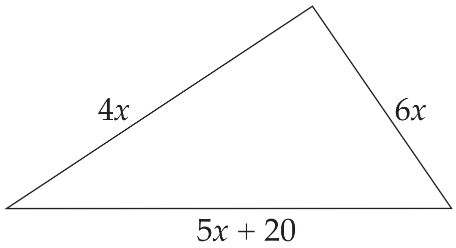
Is the product of the length and width 35 square inches? ________

6. Consider the rectangle and the triangle below. What is the value of x if the area of the rectangle equals the perimeter of the triangle?



Let ____ equal

the area of the rectangle.

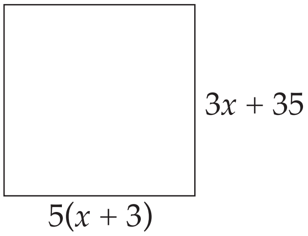


Let ____ equal

the perimeter of the triangle.

Now set up an equation. Let the area of the rectangle equal the perimeter of the triangle. Solve for x.

7. A **square** is a four-sided figure with *all sides equal*. Find the value of x so that the figure is a square.



Circle the letter of the correct answer.

8. Mrs. Jones brings \$142.50 to pay for the family's expenses to see Florida A&M University play football. She has to pay \$10.00 to park. An adult ticket costs \$45.00. She has 4 children who qualify for student tickets. She has \$27.50 left at the end of the day. Which equation can you use to find the cost of a student ticket?

   a. $4x + 45 = \$142.50$

   b. $\$27.50 + 4x + 45 = \$142.50$

   c. $\$142.50 - 10 - 45 - 4x = \$27.50$

   d. $\$142.50 - 10 - 45 + 4x = \$27.50$

Answer the following. Show **essential steps**.

Consecutive **even** *integers* are numbers like 6, 8, and 10 or 14, 16, and 18. Note that you add 2 to the smallest to get the second number and 4 to the smallest to get the third number. Use this information to solve the following problem.

9. The sum of three consecutive even integers is 198. Find the numbers.

Description	**Value**
first number x	=
second number $x + 2$	=
third number $x + 4$	=
sum ______________	= 198

Set up an equation and solve for x. Substitute your answer back into the table above to find all answers. Do the numbers add up to 198?

Practice

Use the list below to complete the following statements.

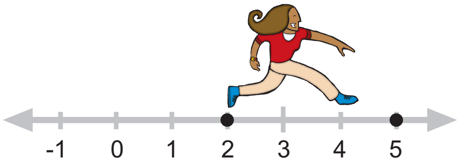
consecutive
even numbers
integers
square
table

1. An orderly display of numerical information in rows and columns is called a ______________________ or chart.
2. A rectangle with four sides the same length is called a ______________________ .
3. Consecutive even ______________________ are numbers like 6, 8, and 10 or 14, 16, and 18.
4. ______________________ means in order.
5. ______________________ are any whole numbers divisible by 2.

Lesson Four Purpose

- Understand the relative size of integers, rational numbers, irrational numbers, and real numbers. (MA.A.1.4.2)
- Understand that numbers can be represented in a variety of equivalent forms, including integers, fractions, decimals, percents, scientific notation, exponents, radicals, and absolute value. (MA.A.1.4.4)
- Describe, analyze, and generalize relationships, patterns, and functions using words, symbols, variables, tables, and graphs. (MA.D.1.4.1)
- Use systems of equations and inequalities to solve real-world problems algebraically. (MA.D.2.4.2)

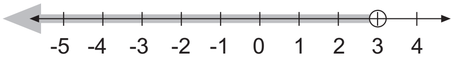
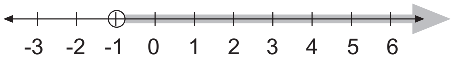
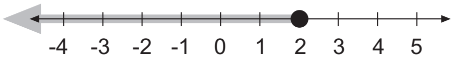
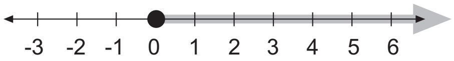
Graphing Inequalities on a Number Line



In Unit 1 we compared numbers using **inequality** symbols. We also solved simple *inequalities* and added numbers using a **number line**. In this unit we will graph inequalities on a number line. A **graph of a number** is the point on a number line paired with the number. Graphing solutions on a number line will help you visualize solutions.

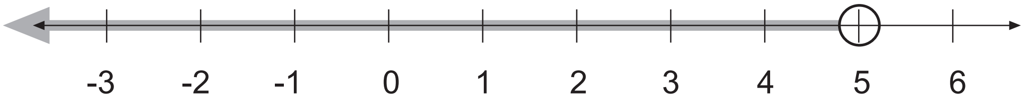
Here are some examples of inequalities, their verbal meanings, and their graphs.

Inequalities

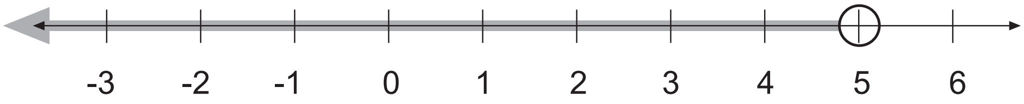
Inequality	Meaning	Graph
a. $x < 3$	All real numbers less than 3.	 Open circle means that 3 is *not* a solution. Shade to left.
b. $x > -1$	All real numbers greater than -1.	 Open circle means that -1 is *not* a solution. Shade to right.
c. $x \leq 2$	All real numbers less than or equal to 2.	 Solid circle means that 2 is a solution. Shade to left.
d. $x \geq 0$	All real numbers greater than or equal to 0.	 Solid circle means that 0 is a solution. Shade to right.

For each example, the inequality is written with the variable on the left. Inequalities can also be written with the variable on the right. However, graphing is easier if the variable is on the left.

Consider $x < 5$, which means the same as $5 > x$. Note that the graph of $x < 5$ is all real numbers less than 5.



The graph of $5 > x$ is all real numbers that 5 is greater than.



To write an inequality that is *equivalent* to (or the same as) $x < 5$, move the number and variable to the opposite side of the inequality, and then reverse the inequality.

$$x < 5$$
$$5 > x$$

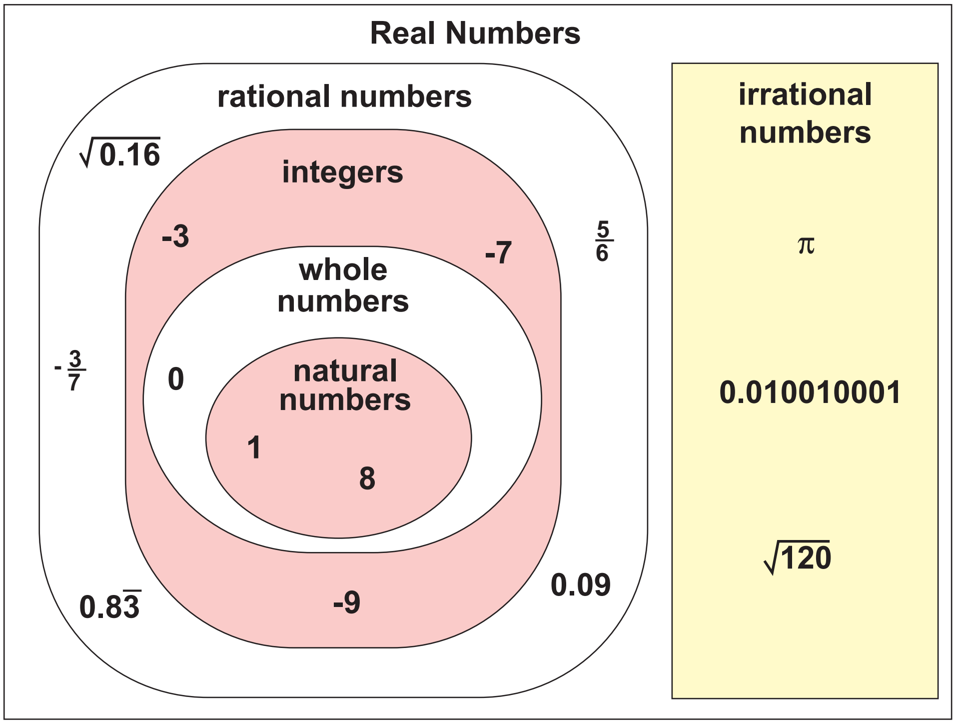
$x < 5$ means the same as

$5 > x$

The inequality $y \geq -2$ is equivalent to $-2 \leq y$. Both inequalities can be written as the *set of all* **real numbers** that are *greater than or equal to* -2.

The inequality $0 \leq x$ is equivalent to $x \geq 0$. Each can be written as the *set of all real numbers* that are *greater than or equal to* zero.

Remember: *Real numbers* are all **rational numbers** and all **irrational numbers**.



Rational numbers can be expressed as a **ratio** of two *integers*.

rational numbers	4	$-3\frac{3}{4}$	0.250	0	$0.33\bar{3}$
expressed as ratio of two integers	$\frac{4}{1}$	$-\frac{15}{4}$	$\frac{1}{4}$	$\frac{0}{1}$	$\frac{1}{3}$

Note: All integers are rational numbers.

A *ratio* is a *quotient* (the result of a division) of two numbers used to compare two quantities. For example, a ratio of 8 to 11 is $\frac{8}{11}$.

Practice

Match each **inequality** *with the correct* **graph**.

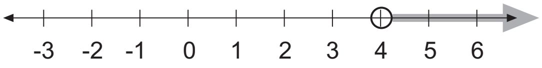
______ 1. $x \geq -3$

______ 2. $x \leq 0$

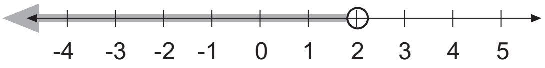
______ 3. $x > 4$

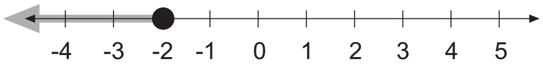
______ 4. $2 > x$

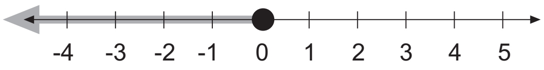
______ 5. $x \leq -2$

a.


b.


c.


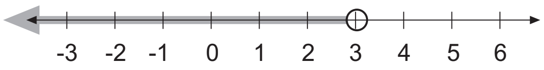
d.


e.


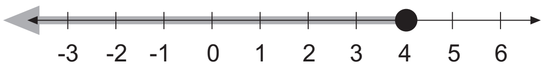
Write an **inequality** *for each* **graph**.

6. ______________________________

7. ______________________________



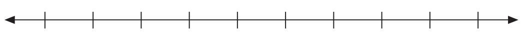
8. ______________________________



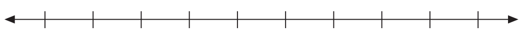
9. ______________________________

Graph each **inequality**.

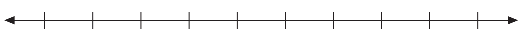
10. $x \geq -1$



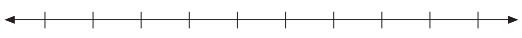
11. $x < 0$



12. $x > 5$



13. $x \leq -3$



Solving Inequalities

We have been solving *equations* since Unit 1. When we solve inequalities, the procedures are the same except for one important difference.

> **When we multiply or divide both sides of an inequality by the same *negative number*, we *reverse* the direction of the inequality symbol.**

Example: Solve by *dividing* by a *negative number* and *reversing* the inequality sign.

$$-3x < 6$$

$$\frac{-3x}{-3} > \frac{6}{-3}$$ divide each side by -3 and reverse the inequality symbol

$$x > -2$$

To check this solution, pick any number *greater than* -2 and substitute your choice into the original inequality. For instance, -1, 0, or 3, or 3,000 could be substituted into the original problem.

Check with different solutions of numbers *greater than* -2:

substitute -1

$$\begin{aligned} -3x &< 6 \\ -3(-1) &< 6 \\ 3 &< 6 \end{aligned}$$ *It checks!*

substitute 3

$$\begin{aligned} -3x &< 6 \\ -3(3) &< 6 \\ -9 &< 6 \end{aligned}$$ *It checks!*

substitute 0

$$\begin{aligned} -3x &< 6 \\ -3(0) &< 6 \\ 0 &< 6 \end{aligned}$$ *It checks!*

substitute 3,000

$$\begin{aligned} -3x &< 6 \\ -3(3{,}000) &< 6 \\ -9{,}000 &< 6 \end{aligned}$$ *It checks!*

Notice that -1, 0, 3, and 3,000 are all *greater than* -2 and each one *checks* as a solution.

Study the following examples.

Example: Solve by *multiplying* by a *negative number* and *reversing* the inequality sign.

$-\frac{1}{3}y \geq 4$

$(-3)\ -\frac{1}{3}y \leq 4(-3)$ multiply each side by -3 and reverse the inequality symbol

$y \leq -12$

Example: Solve by first adding, then *dividing* by a *negative number*, and *reversing* the inequality sign.

$-3a - 4 > 2$

$-3a - 4 + 4 > 2 + 4$ add 4 to each side

$-3a > 6$

$\frac{-3a}{-3} < \frac{6}{-3}$ divide each side by -3 and reverse the inequality symbol

$a < -2$

Example: Solve by first subtracting, then *multiplying* by a *negative number*, and *reversing* the inequality sign.

$\frac{y}{-2} + 5 \leq 0$

$\frac{y}{-2} + 5 - 5 \leq 0 - 5$ subtract 5 from each side

$\frac{y}{-2} \leq -5$

$\frac{(-2)y}{-2} \geq (-5)(-2)$ multiply each side by -2 and reverse the inequality symbol

$y \geq 10$

Example: Solve by first subtracting, then *multiplying* by a *positive number* and **not** *reversing* the inequality sign.

$$\begin{array}{rcll}
\frac{n}{2} + 5 & \leq & 2 & \\
\frac{n}{2} + 5 - 5 & \leq & 2 - 5 & \text{subtract 5 from each side} \\
\frac{n}{2} & \leq & -3 & \\
\frac{(2)n}{2} & \leq & -3(2) & \text{multiply each side by 2, } \textit{but} \\
n & \leq & -6 & \text{do } \textit{not} \text{ reverse the inequality symbol because} \\
 & & & \text{we multiplied by a positive number}
\end{array}$$

When multiplying or dividing both sides of an inequality by the same *positive number*, do *not* reverse the inequality symbol—leave it alone.

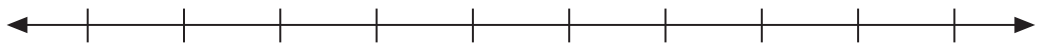
Example: Solve by first adding, then *dividing* by a positive number, and **not** *reversing* the inequality sign.

$$\begin{array}{rcll}
7x - 3 & > & -24 & \\
7x - 3 + 3 & > & -24 + 3 & \text{add 3 to each side} \\
7x & > & -21 & \\
\frac{7x}{7} & > & \frac{-21}{7} & \text{divide each side by 7} \\
x & > & -3 & \text{do } \textit{not} \text{ reverse the inequality symbol because} \\
 & & & \text{we divided by a positive number}
\end{array}$$

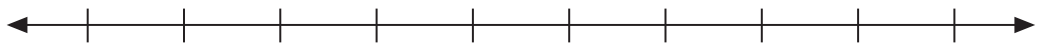
Practice

Solve each **inequality**. *Show* **essential steps**. *Then* **graph the solutions**.

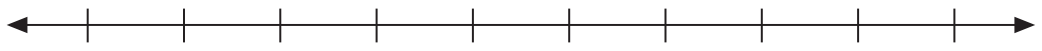
1. $x + 5 \geq 2$



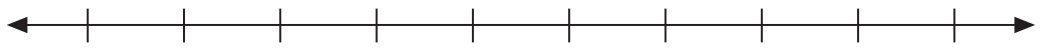
2. $y - 1 \leq 5$



3. $4 < n - 1$

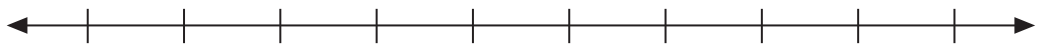


4. $2 \geq y - 4$



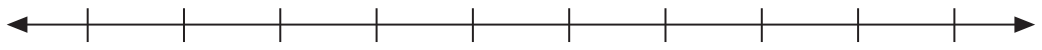
5. $5a - 2 \leq 3$

6. $-\frac{1}{4}y > 0$

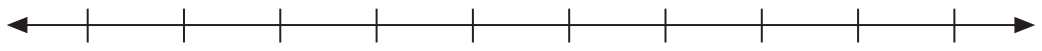


7. $-2a \geq -12$

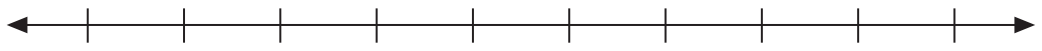
8. $\frac{a}{3} - 3 < 1$



9. $\frac{y}{2} - 6 < -5$



10. $\frac{a}{-3} + 9 < 8$



Practice

Solve *the following*. *Show* **essential steps**.

1. $2y + 1 \leq 4$

2. $-\frac{1a}{3} - 4 > 2$

3. $-11a + 3 < -30$

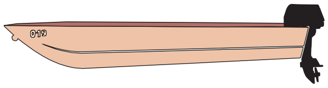
4. $\frac{1}{5}y + 9 \leq 8$

5. $-10 < 2b - 14$

6. $10y + 3 \leq 8$

Study the following.

Many problems in everyday life involve inequalities.



Example: A summer camp needs a boat with a motor. A local civic club will donate the money on the condition that the camp will spend *less than* \$1,500 for both. The camp decides to buy a boat for \$1,050. How much can be spent on the motor?

Choose a variable. *Let* x = cost of the motor,
then let $x + 1{,}050$ = cost of motor and boat,

and cost of motor + cost of boat < total money.
Write as an inequality: $x + 1{,}050 < 1{,}500$
solve $x + 1{,}050 - 1{,}050 < 1{,}500 - 1{,}050$
$x < \$450$

Interpretation of solution: The camp can spend *any* amount *less than* \$450 for the motor. (**Note:** The motor *cannot* cost \$450.)

For the following:

- **choose a variable**
- **set up an inequality**
- **solve**
- **interpret your solution**.

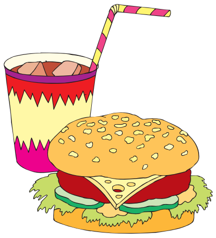
7. If $50 is added to 2 times the amount of money in a bank, the result is less than $150. What is the greatest amount of money that could be in the bank?

Interpretation of solution: ______________________________

____

____

8. Hamburgers cost $2.50 and a soft drink is $1.50. If you want to buy one soft drink, what is the greatest number of hamburgers you could also buy and spend less than $10.00?



Interpretation of solution: ______________________________

____

____

9. Annie baby-sits on Friday nights and Saturdays for $3.00 an hour. Find the fewest number of hours she can baby-sit and earn more than $20.00 a week.

Interpretation of solution: ________________________________

____

____

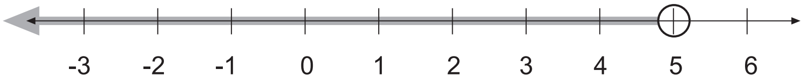
Practice

Write **True** *if the statement is correct. Write* **False** *if the statement is* not *correct.*

__________ 1. Graphing solutions on a number line will help you visualize solutions.

__________ 2. An inequality can only be written with the variable on the right.

__________ 3. The graph below of $x < 5$ is all real numbers greater than 5.



__________ 4. *Real numbers* are all rational and irrational numbers.

__________ 5. A *ratio* is a quotient of two numbers used to compare two quantities.

__________ 6. To write an inequality that is equivalent to $x < 5$, move the number and variable to the opposite side of the inequality, and then reverse the inequality.

__________ 7. When we multiply or divide each side of an inequality by the same negative number, we reverse the direction of the inequality symbol.

__________ 8. There are no problems in everyday life that involve inequalities.

__________ 9. An *inequality* is a sentence that states one expression is greater than, greater than or equal to, less than, less than or equal to, or not equal to another expression.

Practice

Use the list below to write the correct term for each definition on the line provided.

decrease
difference
equation
increase
reciprocals
simplify an expression
solve
sum

____________________ 1. the result of a subtraction

____________________ 2. to find all numbers that make an equation or inequality true

____________________ 3. to make less

____________________ 4. a mathematical sentence that equates one expression to another expression

____________________ 5. to make greater

____________________ 6. two numbers whose product is 1

____________________ 7. the result of an addition

____________________ 8. to perform as many of the indicated operations as possible

Practice

Match each definition with the correct term. Write the letter on the line provided.

______ 1. a sentence that states one expression is greater than, greater than or equal to, less than, less than or equal to, or not equal to another expression

______ 2. the length of the boundary around a figure

______ 3. the point on a number line paired with the number

______ 4. a one-dimensional measure that is the measurable property of line segments

______ 5. an orderly display of numerical information in rows and columns

______ 6. the number of degrees (°) of an angle

______ 7. a polygon with three sides

______ 8. the shape made by two rays extending from a common endpoint, the vertex

______ 9. any number *not* divisible by 2

______ 10. a parallelogram with four right angles

A. angle

B. graph (of a number)

C. inequality

D. length (*l*)

E. measure (m) of an angle ($\angle$)

F. odd numbers

G. perimeter (*P*)

H. rectangle

I. table

J. triangle

Unit Review

Solve these equations. *Show all* **essential steps**.

1. $4y + 2 = 30$

2. $-5x - 6 = 34$

3. $\frac{x}{3} + 7 = -3$

4. $\frac{x}{-4} - 2 = 10$

5. $\frac{1x}{6} + 2 = 8$

6. $\frac{-3x}{4} - 8 = -2$

7. $5 - x = 12$

8. $12 = -7 - x$

9. $8 - \frac{2x}{3} = 12$

10. What is the reciprocal of $-\frac{3}{4}$? ____________

Number 11 is a **gridded-response item**.
Write answer along the top of the grid and correctly mark it below.

11. What is the reciprocal of 8?

Mark your answer on the grid to the right.

	⊘	⊘	⊘	
⊙	⊙	⊙	⊙	⊙
⓪	⓪	⓪	⓪	⓪
①	①	①	①	①
②	②	②	②	②
③	③	③	③	③
④	④	④	④	④
⑤	⑤	⑤	⑤	⑤
⑥	⑥	⑥	⑥	⑥
⑦	⑦	⑦	⑦	⑦
⑧	⑧	⑧	⑧	⑧
⑨	⑨	⑨	⑨	⑨

Simplify *the following.*

12. $-5(x + 2) + 16$

13. $15 + 2(x + 8)$

14. $5x - 7x$

15. $-8x - 14 + 10x - 20$

Solve these equations. *Show* **essential steps**.

16. $7x + 3 - 8x + 12 = -6$

17. $7x + 3(x + 2) = 36$

18. $-\frac{1}{2}(x + 10) = -15$

19. $5x - 8 = 4x + 10$

20. $-8(1 - 2x) = 5(2x - 6)$

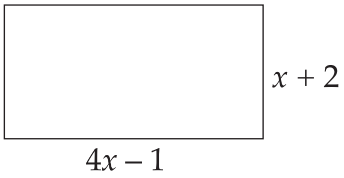
Write an **equation** *and* **solve for** ***x***.

21. The *sum* of 2x and 7 equals 19. What is the number?

22. $\frac{1}{2}$ of x *decreased* by 7 is -10. What is the number?

23. The *difference* between 14 and 2x is -10. What is the number?

24. The *perimeter* is the sum of the lengths of the sides of a figure. The perimeter of the rectangle below is 52. Write an equation and solve for x.



Answer the following. Show **essential steps**.

Consecutive **odd** integers are numbers like 3, 5, and 7 or 15, 17, and 19. Note that you add 2 to the smallest to get the second number and 4 to the smallest to get to the third number. Use this information to solve the following problem.

25. The sum of three consecutive *odd* integers is 159.

Description		**Value**
first number	x	=
second number	$x + 2$	=
third number	$x + 4$	=
sum	________	= 159

Set up an equation and solve for x. Substitute your answer back into the table to find all answers. Do the numbers add up to 159?

Match each **equation** *with its* **graph**.

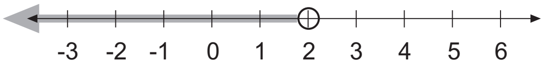
______ 26. $x \geq 2$

______ 27. $x < 2$

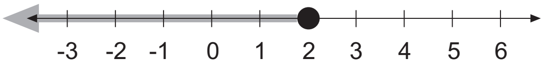
______ 28. $2 \geq x$

______ 29. $x < -2$

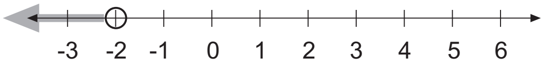
a.



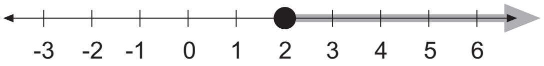
b.



c.



d.



Solve and graph.

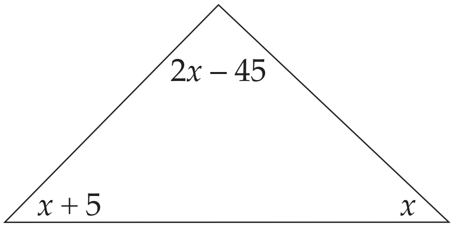
30. $-5x + 6 > -34$

31. $\frac{x}{-2} + 6 \leq 0$

Answer the following.

Bonus Problems:

32. The sum of the measures of the angles in any triangle is 180 degrees. Find x, and then find the *measure* of each angle for the triangle below.



33. Solve and graph this inequality.

$-20 < -2x - 14$

Unit 4: Geometry and Spatial Sense

This unit emphasizes the use and meaning of many words and properties of geometry.

Unit Focus

Number Sense, Concepts, and Operations

- Understand concrete and symbolic representations of real numbers in real-world situations. (MA.A.1.4.3)
- Understand that numbers can be represented in a variety of equivalent forms, including integers, fractions, decimals, percents, scientific notation, exponents, radicals, and absolute value. (MA.A.1.4.4)
- Add, subtract, multiply, and divide real numbers, including square roots and exponents, using appropriate methods of computing, such as mental mathematics, paper and pencil, and calculator. (MA.A.3.4.3)
- Use estimation strategies in complex situations to predict results and to check the reasonableness of results. (MA.A.4.4.1)

Measurement

- Use concrete and graphic models to derive formulas for finding perimeter, area, surface area, circumference, and volume of two- and three-dimensional shapes, including rectangular solids, cylinders, cones, and pyramids. (MA.B.1.4.1)
- Use concrete and graphic models to derive formulas for finding angle measures. (MA.B.1.4.2)
- Relate the concepts of measurement to similarity and proportionality in real-world situations. (MA.B.1.4.3)
- Solve real-world and mathematical problems, involving estimates of measurements, including length, perimeter,

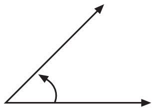
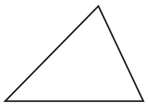
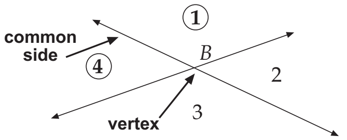
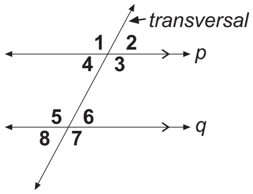
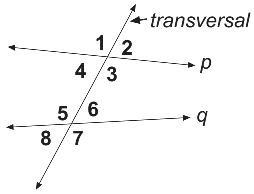
area, and volume and estimate the effects of measurement errors on calculations. (MA.B.3.4.1)

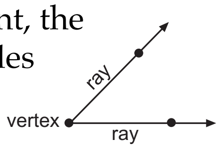
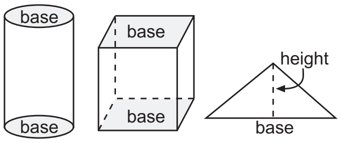
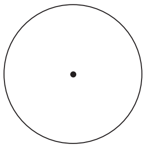
Geometry and Spatial Relations

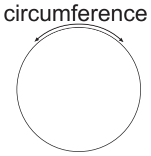
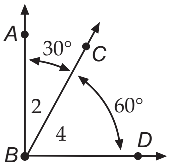
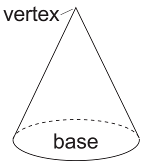
- Understand geometric concepts such as perpendicularity, parallelism, congruency, and similarity. (MA.C.2.4.1)
- Represent and apply geometric properties and relationships to solve real-world and mathematical problems including ratio and proportion. (MA.C.3.4.1)
- Using a rectangular coordinate system (graph), apply and algebraically verify properties of two-dimensional figures. (MA.C.3.4.2)

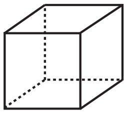
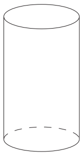
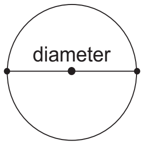
Vocabulary

Use the vocabulary words and definitions below as a reference for this unit.

acute angle	an angle with a measure of less than 90°	
acute triangle	a triangle with three acute angles	
adjacent angles	two angles having a common vertex and sharing a common side	 Angle 1 and angle 4 have a common side and also the same vertex. They are adjacent angles because they are next to each other.
alternate angles	a pair of angles that lie on opposite sides and at opposite ends of a transversal  Alternate angles are equal when the lines intersected by a transversal are parallel.  Even when lines cut by a transversal are *not* parallel, we still use the same vocabulary. alternate exterior angles are angles whose points lie on the opposite sides of a transversal line and on the *outside* of the lines it intersects **∠ 1 and ∠ 7** **∠ 2 and ∠ 8** alternate interior angles are angles whose points lie on the opposite sides of a transversal line and on the *inside* of the lines it intersects **∠ 3 and ∠ 5** **∠ 4 and ∠ 6**	

altitude	see *height*
angle (∠)	the shape made by two rays extending from a common endpoint, the vertex; measures of angles are described in degrees (°) 
area (*A*)	the inside region of a two-dimensional figure measured in square units *Example*: A rectangle with sides of four units by six units contains 24 square units or has an area of 24 square units.
associative property	the way in which three or more numbers are grouped for addition or multiplication does *not* change their sum or product *Example*: (5 + 6) + 9 = 5 + (6 + 9) *or* (2 x 3) x 8 = 2 x (3 x 8)
axes (of a graph)	the horizontal and vertical number lines used in a rectangular graph or coordinate grid system as a fixed reference for determining the position of a point; (singular: *axis*)
base (*b*)	the line or plane upon which a figure is thought of as resting 
center (of a circle)	the point from which all points on the circle are the same distance 
circle	the set of all points in a plane that are all the same distance from a given point called the center

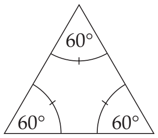
circumference (*C*)	the perimeter of a circle; the distance around a circle 
commutative property	the order in which any two numbers are added or multiplied does *not* change their sum or product *Example*: 2 + 3 = 3 + 2 *or* 4 x 7 = 7 x 4
complementary angles	two angles, the sum of which is exactly 90°  $m\angle ABC + m\angle CBD = 90°$ complementary angles
cone	a three-dimensional figure with one circular base in which a curved surface connects the base to the vertex 
congruent (≅)	figures or objects that are the same shape and the same size
coordinate grid or system	network of evenly spaced, parallel horizontal and vertical lines especially designed for locating points, displaying data, or drawing maps
coordinates	numbers that correspond to points on a graph in the form (x, y)
corresponding angles	a pair of angles that are in matching positions and lie on the same side of a transversal
corresponding angles and sides	the matching angles and sides in similar figures

cross product	the product of one numerator and the opposite denominator in a pair of fractions *Example*: Is $\frac{2}{5}$ equal to $\frac{6}{15}$? $\frac{2}{5} \overset{?}{=} \frac{6}{15}$ 2 x 15 $\overset{?}{=}$ 5 x 6 The cross products are 2 x 15 and 5 x 6 $30 = 30$ Both cross products equal 30. The cross products of equivalent fractions are equal. Yes, $\frac{2}{5} = \frac{6}{15}$.
cube	a rectangular prism that has six square faces 
cubic units	units for measuring volume
cylinder	a three-dimensional figure with two parallel congruent circular bases *Example*: a can 
degree (°)	common unit used in measuring angles
diameter (*d*)	a line segment from any point on the circle passing through the center to another point on the circle 
distributive property	for any real numbers a, b, and x, $x(a + b) = ax + bx$

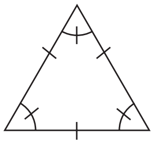
endpoint .... either of two points marking the end of a line segment

equation .... a mathematical sentence that equates one expression to another expression
Example: $2x = 10$

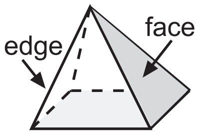
equiangular triangle .... a triangle with three equal angles



equilateral triangle .... a triangle with three congruent sides and three congruent angles



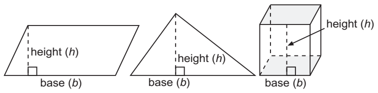
face .... one of the plane surfaces bounding a three-dimensional figure



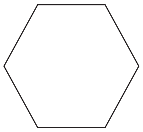
formula .... a way of expressing a relationship using variables or symbols that represent numbers

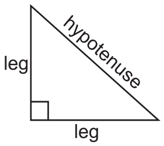
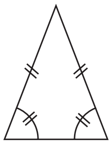
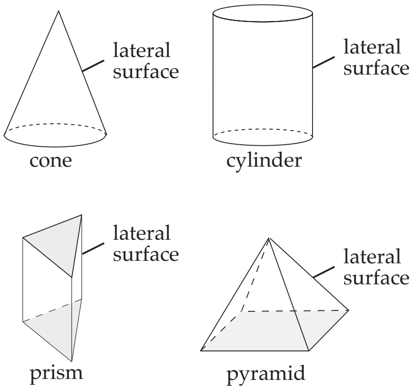
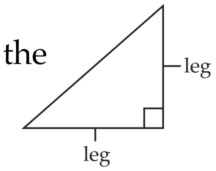
graph of a point .... the point assigned to an ordered pair on a coordinate plane

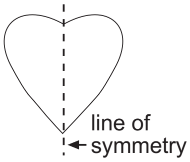
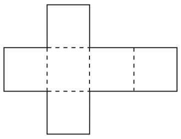
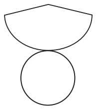
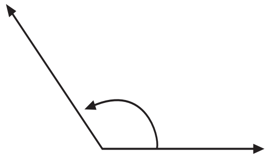
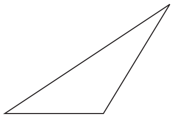
height (*h*) .... a line segment extending from the vertex or *apex* (highest point) of a figure to its base and forming a right angle with the base or basal plane

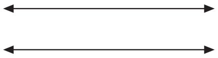
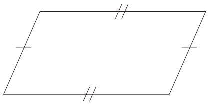
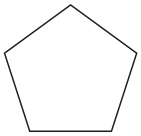
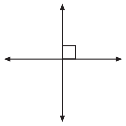


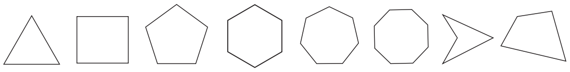
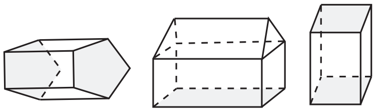
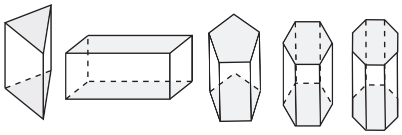
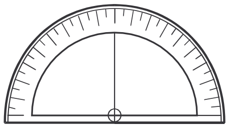
hexagon .... a polygon with six sides

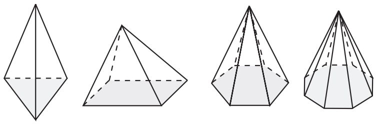
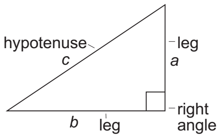
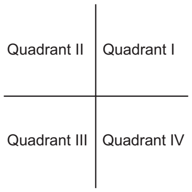
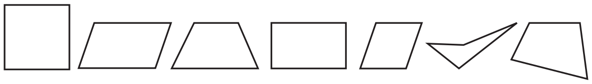
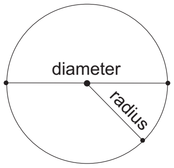


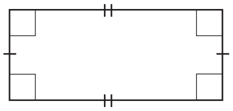
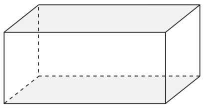
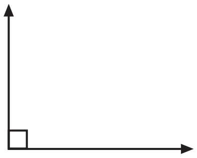
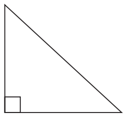
hypotenuse	the longest side of a right triangle; the side opposite the right angle in a right triangle 
intersect	to meet or cross at one point
intersection	the point at which two lines meet
isosceles triangle	a triangle with at least two congruent sides and two congruent angles 
lateral	a surface on the side of a geometric figure, as opposed to the base 
leg	in a right triangle, one of the two sides that form the right angle 
length (*l*)	a one-dimensional measure that is the measurable property of line segments
line ($\longleftrightarrow$)	a straight line that is endless in length 

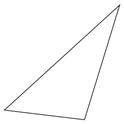
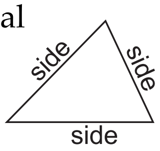
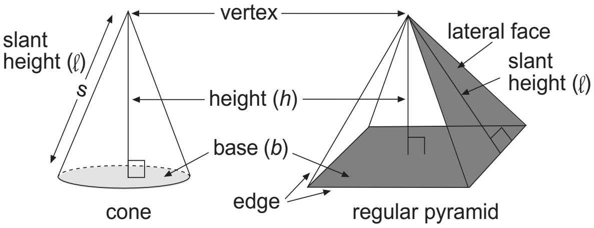
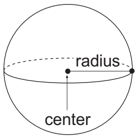
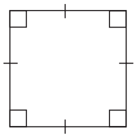
line of symmetry	a line that divides a figure into two congruent halves that are mirror images of each other 
line segment (—)	a portion of a line that has a defined beginning and end  *Example*: The line segment AB is between point A and point B and includes point A and point B.
measure (m) of an angle (∠)	the number of degrees (°) of an angle
net	a plan which can be used to make a model of a solid; a two-dimensional shape that can be folded into a three-dimensional figure  net of a cube  net of cone
obtuse angle	an angle with a measure of more than 90° but less than 180° 
obtuse triangle	a triangle with one obtuse angle 
ordered pair	the location of a single point on a rectangular coordinate system where the digits represent the position relative to the x-axis and y-axis *Example*: (x, y) or $(3, 4)$

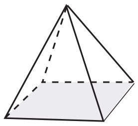
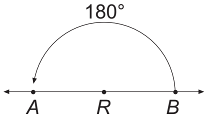
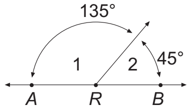
origin	the graph of zero (0) on the number line *or* the intersection of the x-axis and y-axis in a coordinate plane, described by the ordered pair (0, 0)
parallel (‖)	being an equal distance at every point so as to never intersect
parallel lines	two lines in the same plane that never meet; also, lines with equal slopes 
parallelogram	a quadrilateral with two pairs of parallel sides 
pentagon	a polygon with five sides 
perimeter (*P*)	the length of the boundary around a figure; the distance around a polygon
perpendicular (⊥)	forming a right angle
perpendicular lines	two lines that intersect to form right angles 
pi (π)	the symbol designating the ratio of the circumference of a circle to its diameter, with an approximate value of either 3.14 or $\frac{22}{7}$
plane	an undefined, two-dimensional (no depth) geometric surface that has no boundaries specified; a flat surface

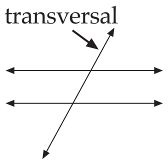
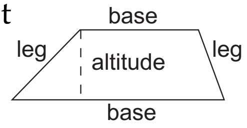
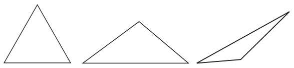
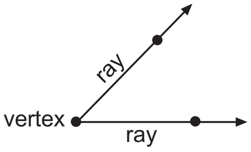
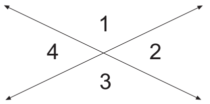
point	a location in space that has no length or width
polygon	a closed plane figure whose sides are straight and do not cross *Example*: triangle (3 sides), quadrilateral (4 sides), pentagon (5 sides), hexagon (6 sides), heptagon (7 sides), octagon (8 sides); concave, convex 
polyhedron	a three-dimensional figure in which all surfaces are polygons 
prism	a three-dimensional figure (polyhedron) with congruent, polygonal bases and lateral faces that are all parallelograms 
proportion	a mathematical sentence stating that two ratios are equal *Example*: The ratio of 1 to 4 equals 25 to 100, that is $\frac{1}{4} = \frac{25}{100}$.
protractor	an instrument used for measuring and drawing angles 

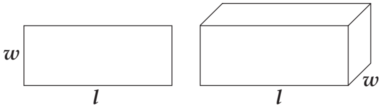
pyramid	a three-dimensional figure (polyhedron) with a single base that is a polygon and whose faces are triangles and meet at a common point (vertex) 
Pythagorean theorem	the square of the hypotenuse (*c*) of a right triangle is equal to the sum of the squares of the legs (*a* and *b*) *Example*: $a^2 + b^2 = c^2$ 
quadrant	any of four regions formed by the axes in a rectangular coordinate system 
quadrilateral	polygon with four sides *Example*: square, parallelogram, trapezoid, rectangle, rhombus, concave quadrilateral, convex quadrilateral 
radius (*r*)	a line segment extending from the center of a circle or sphere to a point on the circle or sphere; (plural: *radii*) 
ratio	the quotient of two numbers used to compare two quantities *Example*: The ratio of 3 to 4 is $\frac{3}{4}$.

ray (→)	a portion of a line that begins at a point and goes on forever in one direction	
rectangle	a parallelogram with four right angles	
rectangular prism	a six-sided prism whose faces are all rectangular *Example*: a brick	
right angle	an angle whose measure is exactly 90°	
right triangle	a triangle with one right angle	
rounded number	a number approximated to a specified place *Example*: A commonly used rule to round a number is as follows. • If the digit in the first place after the specified place is 5 or more, *round up* by adding 1 to the digit in the specified place ($\overset{\curvearrowleft}{\underline{4}6}1$ rounded to the nearest hundred is 500). • If the digit in the first place after the specified place is less than 5, *round down* by *not* changing the digit in the specified place ($\overset{\curvearrowleft}{\underline{4}4}1$ rounded to the nearest hundred is 400).	
scale factor	the ratio between the lengths of corresponding sides of two similar figures	

scalene triangle	a triangle with no congruent sides 
side	the edge of a two-dimensional geometric figure *Example*: A triangle has three sides. 
similar figures	figures that have the same shape but not necessarily the same size
slant height (ℓ)	the shortest distance from the vertex of a right circular cone to the edge of its base; the perpendicular distance from the vertex of a pyramid to one edge of its base 
solid figures	three-dimensional figures that completely enclose a portion of space *Example*: rectangular solid and a sphere
sphere	a three-dimensional figure in which all points on the surface are the same distance from the center 
square	a rectangle with four sides the same length 

square (of a number)	the result when a number is multiplied by itself or used as a factor twice *Example*: 25 is the square of 5.
square pyramid	a pyramid with a square base and four faces that are triangular 
square units	units for measuring area; the measure of the amount of an area that covers a surface
straight angle	an angle whose measure is exactly 180°  m∠ARB = 180° straight angle
substitute	to replace a variable with a numeral *Example*: $8a + 3$ $8 \bullet 5 + 3$
sum	the result of an addition *Example*: In $6 + 8 = 14$, 14 is the sum.
supplementary angles	two angles, the sum of which is exactly 180°  m∠1 + m∠2 = 180° supplementary angles
surface area (*S.A.*) (of a geometric solid)	the sum of the areas of the faces of the figure that create the geometric solid
three-dimensional (3-dimensional)	existing in three dimensions; having length, width, and height

transversal	a line that intersects two or more other (usually parallel) lines in the same plane 
trapezoid	a quadrilateral with just one pair of opposite sides parallel 
triangle	a polygon with three sides; the sum of the measures of the angles is 180° 
two-dimensional (2-dimensional)	existing in two dimensions; having length and width
variable	any symbol that could represent a number
vertex	the common endpoint from which two rays begin or the point where two lines intersect; the point on a triangle or pyramid opposite to and farthest from the base; (plural: *vertices*); vertices are named clockwise or counterclockwise 
vertical angles	the opposite angles formed when two lines intersect  ∠1 and ∠3 are vertical angles ∠2 and ∠4 are also vertical angles

volume (*V*)	the amount of space occupied in three dimensions and expressed in cubic units *Example*: Both capacity and volume are used to measure empty spaces; however, *capacity* usually refers to *fluids*, whereas *volume* usually refers to *solids*.
width (*w*)	a one-dimensional measure of something side to side 
***x*-axis**	the horizontal (⟷) axis on a coordinate plane
***x*-coordinate**	the first number of an ordered pair
***y*-axis**	the vertical (↕) axis on a coordinate plane
***y*-coordinate**	the second number of an ordered pair

Unit 4: Geometry and Spatial Sense

Introduction

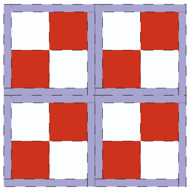
The word geometry means *earth-measuring*. Use of the Pythagorean theorem by the Egyptians to survey land dates back to around 2000 B.C.

In studying geometry, students can quickly see numerous applications for real-life shapes, design, and sculpture. We see parallel lines any time we travel on a highway or park in a marked parking area. Contractors, architects, artists, and engineers use geometry extensively. We see the results of their talents in buildings of every imaginable size and shape. Highways, bridges, and tunnels are also created by people who are experts in geometry.

Contractors, architects, artists, and engineers use geometry extensively.

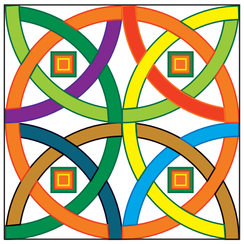
You will learn to identify various types of polygons in this unit. When you look around your home, you will usually find many geometric shapes in upholstery fabrics, curtains, bedspreads, floor coverings, furniture, and artwork.

fabrics



bedspreads

floor coverings



artwork

Lesson One Purpose

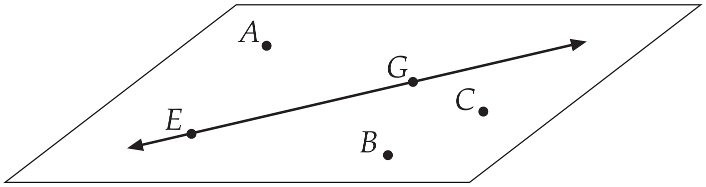
- Understand concrete and symbolic representations of real numbers in real-world situations. (MA.A.1.4.3)
- Use estimation strategies in complex situations to predict results and to check the reasonableness of results. (MA.A.4.4.1)

- Use concrete and graphic models to derive formulas for finding angle measures. (MA.B.1.4.2)
- Understand geometric concepts such as perpendicularity, parallelism, congruency, and similarity. (MA.C.2.4.1)

Geometric Basics

Basic items include **planes**, **points**, and **lines (↔)**.

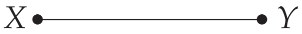
The figure below represents a *plane*—an undefined, two-dimensional geometric surface with no specified boundaries. A plane is a flat surface. We use dots and capital letters to indicate *points*—a location in space that has no **length (l)** or **width (w)**. See points A, B, and C.



plane

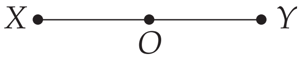
Lines are drawn with arrows at both ends to indicate that the lines are infinitely long. Note how the symbol (↔) drawn over two capital letters is used to describe a line. See line EG ($\overleftrightarrow{EG}$) above.

See a representation of **line segment (—)** XY ($\overline{XY}$) below. $\overline{XY}$ indicates it is a segment with ends at points X and Y. Note how the symbol (—) drawn over two capital letters is used to describe a line segment. The symbol has no arrows because the line segment has a defined beginning and end called **endpoints**. *Endpoints* are either of the two points marking the end of a line segment. X and Y are endpoints of the line segment XY ($\overline{XY}$).



Line segments have measurable *length*. However, it is impossible to measure the lengths of lines because they extend forever in opposite directions.

Suppose that $\overline{XY}$ below has a point O placed on the segment so that the distance from X to O is the same as the distance from O to Y.



We describe the length of $\overline{XO}$ as 6. We write $\overline{XO} = 6$. Likewise the length of $\overline{OY} = 6$. Since the segments have the same length, we can say that the segments are **congruent** (≅), the same shape and the same size.

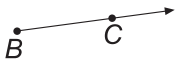
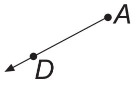
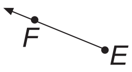
$$\overline{XO} \cong \overline{OY}$$

The symbol ≅ is read: segment XO is *congruent* to segment OY.

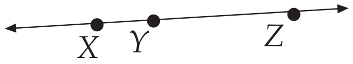
We also draw **rays** (→), a portion of a line that begins at a point and goes on forever in one direction. Rays can be used to construct **angles (∠)**. *Angles* are shapes made by two rays extending from a common endpoint. *Rays* can also be part of lines.

Note how the symbol → drawn over two capital letters is used to describe a ray. Rays are always named with the *endpoint* listed *first* to describe the *originating* (beginning) point of the ray. For example, see the chart below. In ray BC ($\overrightarrow{BC}$), B is the *endpoint* and is listed first.

Name of Rays

Drawing	Name	Endpoint
Ray *BC* 	$\overrightarrow{BC}$	*B*
Ray *AD* 	$\overrightarrow{AD}$	*A*
Ray *EF* 	$\overrightarrow{EF}$	*E*

The line to the right can be labeled many ways:



$\overleftrightarrow{XZ}$ or $\overleftrightarrow{ZX}$
$\overleftrightarrow{XY}$ or $\overleftrightarrow{YX}$
$\overleftrightarrow{YZ}$ or $\overleftrightarrow{ZY}$

The line also contains rays. Some possibilities are as follows:

$\overrightarrow{XY}$ and $\overrightarrow{XZ}$ name the same ray.

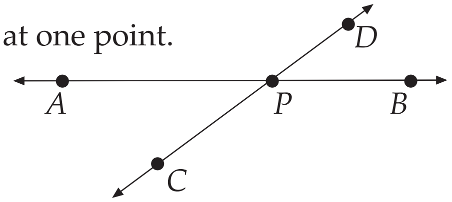
$\overrightarrow{YX}$ and $\overrightarrow{YZ}$ are opposite rays.

$\overrightarrow{XY}$ and $\overrightarrow{YZ}$ are distinctly different rays.

$\overrightarrow{ZY}$ and $\overrightarrow{XY}$ also are different rays.

Note: $\overrightarrow{XZ}$ and $\overrightarrow{ZX}$ are different rays but are not opposite rays because they overlap and do *not* have the same endpoint. $\overrightarrow{YX}$ and $\overrightarrow{YZ}$ are opposite rays because they share the same endpoint and do *not* overlap.

When lines **intersect**, they meet or cross at one point.



$\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ intersect at point P.

We can also identify segments that lie on $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$. For example, $\overline{AP}$ lies on $\overleftrightarrow{AB}$.

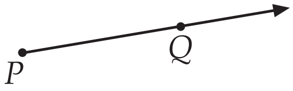
The same segment $\overline{AB}$ can be written as $\overline{BA}$. $\overline{CP}$ (or $\overline{PC}$) also lies on $\overleftrightarrow{CD}$.

Note: Appendix B contains a list of mathematical symbols and their meanings.

Practice

Use the following to write the **symbol** *and to name the* **endpoint(s)** *of each figure.*

Symbols **Endpoint(s)**

1. Ray: 

__________ __________

2. Segment: 

__________ __________

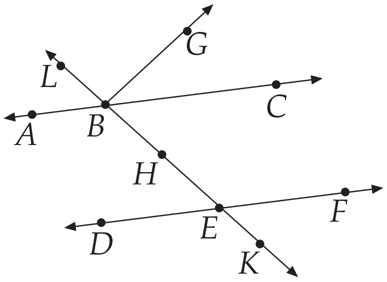
3. Line: 

__________ __________

4. Ray: 

__________ __________

Answer the following using the figure below.



5. Write four names for line *BK*. ______________________________

____

6. Name three different line segments that lie on $\overleftrightarrow{BK}$. ____________

____

7. Name two lines that intersect. ______________________________

____

8. Name five rays that have the same endpoint. ________________

____

____

9. Are $\overrightarrow{DE}$ and $\overrightarrow{ED}$ the same ray? ____________________________

Explain. ____

____

10. If $\overline{BH}$ and $\overline{HE}$ have the same lengths, are they congruent? ______

____

11. Could $\overleftrightarrow{AC}$ and $\overleftrightarrow{DF}$ be congruent? __________________________

Explain. ____

____

12. Could $\overline{AC}$ and $\overline{DF}$ be congruent? __________________________

Explain. ____

____

13. Name opposite rays that have endpoint E. ___________________

____

14. Are $\overrightarrow{DE}$ and $\overrightarrow{FE}$ opposite rays? ___________________________

Explain. ____

____

15. Are $\overline{AB}$ and $\overline{BA}$ the same line segment? ______________________

Explain. ____

____

16. Are $\overrightarrow{AB}$ and $\overrightarrow{BA}$ the same ray? ______________________________

Explain. ____

____

Practice

Use the list below to write the correct term for each definition on the line provided.

angle
congruent (≅)
endpoint
intersect
length (*l*)
line
line segment
plane
point
ray
width (*w*)

________________________ 1. a location in space that has no length or width

________________________ 2. an undefined, two-dimensional (no depth) geometric surface that has no boundaries specified; a flat surface

________________________ 3. a portion of a line that begins at a point and goes on forever in one direction

________________________ 4. a one-dimensional measure that is the measurable property of line segments

________________________ 5. figures or objects that are the same shape and the same size

________________________ 6. a straight line that is endless in length

________________________ 7. either of two points marking the end of a line segment

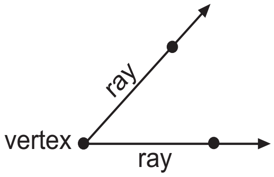
________________________ 8. a portion of a line that has a defined beginning and end

________________________ 9. to meet or cross at one point

__________________________ 10. a one-dimensional measure of something side to side

__________________________ 11. the shape made by two rays extending from a common endpoint

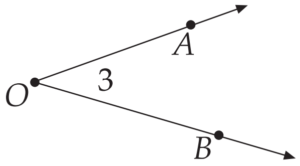
Measuring and Classifying Angles



The *sides* of an angle are formed by two rays (→) extending from a common endpoint called the **vertex**.

Naming an Angle

Consider the following figure.



Remember: The symbol $\angle$ indicates an angle.

You can name an angle in three ways.

- use a three-letter name in this order: point on one ray; vertex; point on other ray, such as $\angle AOB$ *or* $\angle BOA$
- use a one-letter name: vertex, if there is only one angle with this vertex in the diagram, such as $\angle O$
- use a numerical name if the number is within the rays of the angle, such as $\angle 3$

The angle is formed by rays $\overrightarrow{OA}$ and $\overrightarrow{OB}$. The rays are portions of lines that begin at a point and go on forever in one direction.

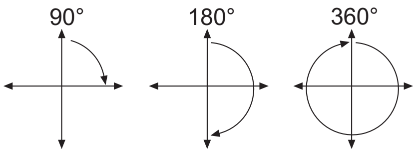
The point O, which is the same endpoint for $\overrightarrow{OA}$ and $\overrightarrow{OB}$, is the *vertex* of angle AOB. When using three letters to name an angle, the *vertex* letter is listed in the middle.

The measure of $\angle O$ is written as $m\angle O$. Sometimes two (or more) angles have the same measure. When two angles have the same measure, they are congruent.

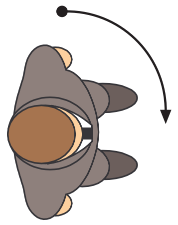
Measuring an Angle

The **measure (m) of an angle (∠)** is described in **degrees (°)**. When you turn around and face backward, you could say you "did a 180." If you turn all the way around, it is a 360. An angle is a turn around a point. The size of an angle is the measure of how far one side has turned from the other side.

0° = no turn
90° = right-angle turn
180° = straight backward
360° = back facing forward again



0° = no turn



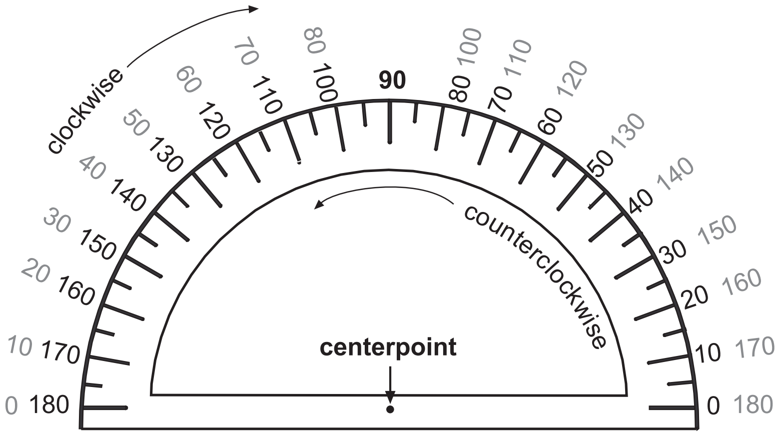
90° = right-angle turn

180° = straight backward

360° = back facing forward again

Using Protractors to Measure Angles

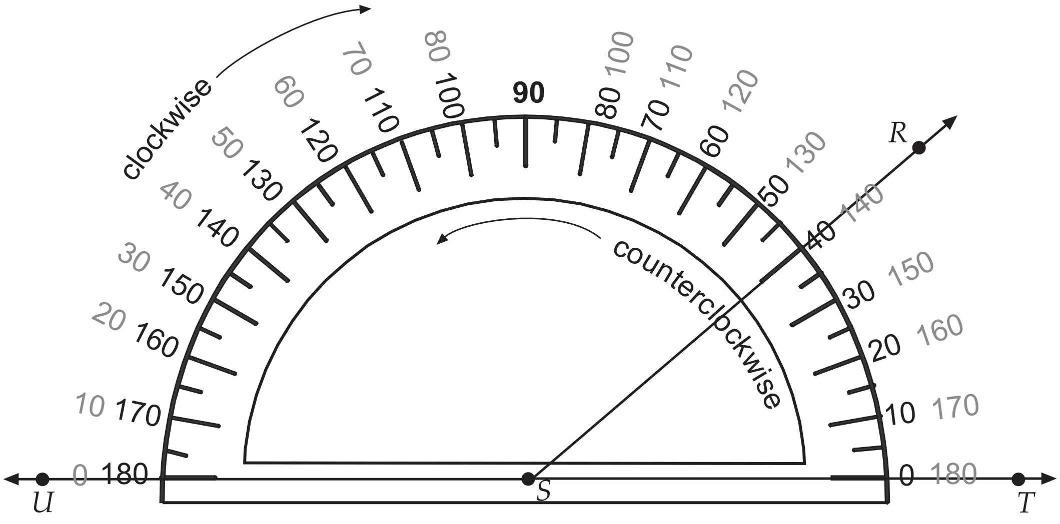
Protractors are marked from 0 to 180 degrees in both a clockwise manner and a counterclockwise manner. We see 10 and 170 in the same position. We see 55 and 125 in the same position. If we estimate the size of the angle before using the protractor, there is no doubt which measure is correct.



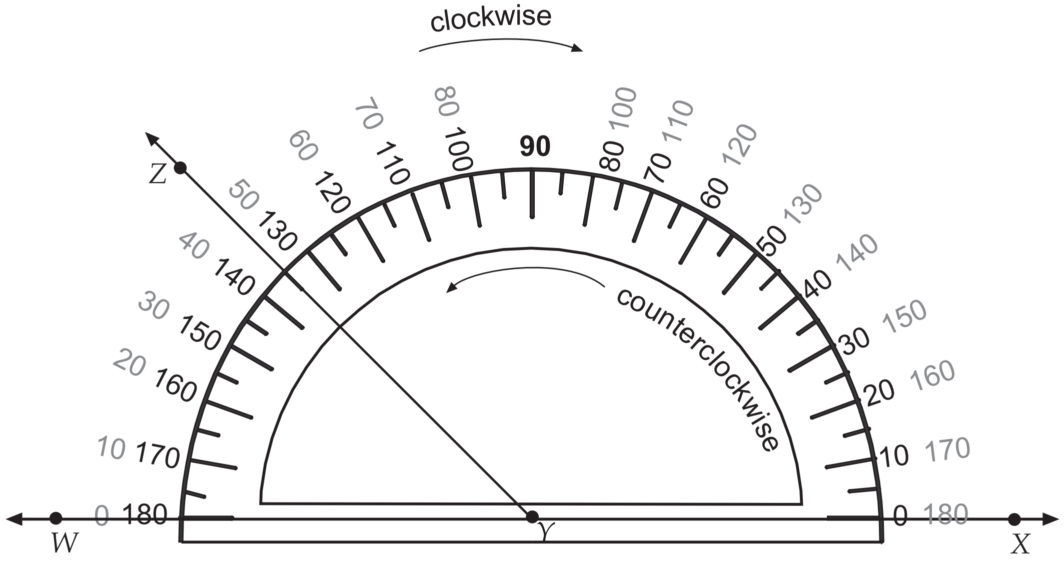
When using a protractor, make sure the vertex is lined up correctly and that one *ray* (→) passes through the zero measure. A straightedge is often helpful to extend a ray for easier reading of the measure.

A *protractor* is used to measure angles. Follow these steps to use a protractor.

1. Place the centerpoint of the protractor on the vertex of the angle.
2. Line up the protractor's 0 degree line with one side of the angle.
3. Read the measure of the angle where the other side crosses the protractor.



- The measurement of $\angle TSR$ is 40° .
  40° is read 40 degrees.
- The measurement of $\angle USR$ is 140° .



- The measurement of ∠*XYZ* is 135° .
- The measurement of ∠*WYZ* is 45° .

Naming Different Size Angles

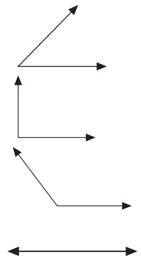
Angles are named for the way they relate to 90 degrees and 180 degrees.

acute angle = < 90°

right angle = 90°

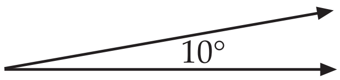
obtuse angle = > 90° and < 180°

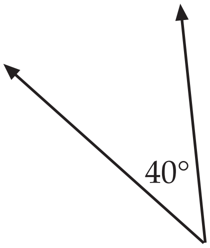
straight angle = 180°

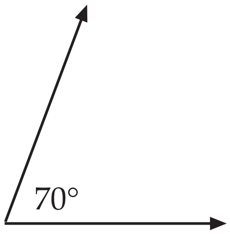


An *acute angle* measures greater than 0° but less than 90°.

Examples:

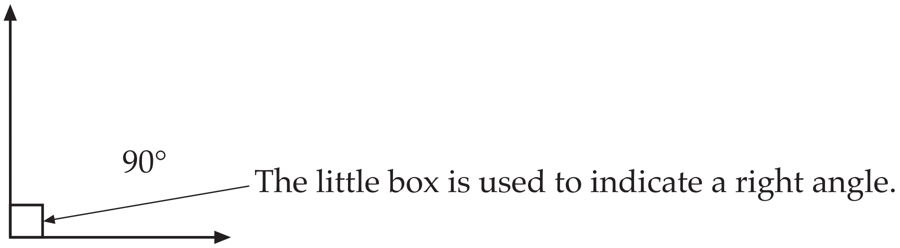






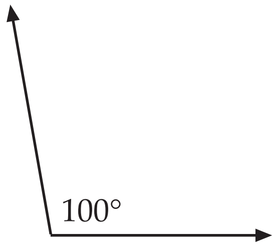
A *right angle* measures exactly 90°.

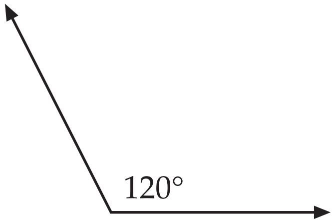
Example:



An *obtuse angle* measures greater than 90° but less than 180°.

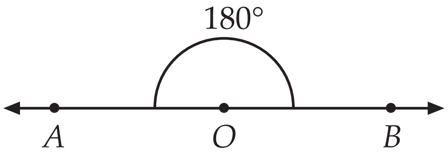
Examples:





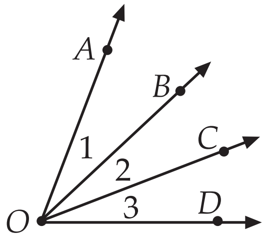
A *straight angle* measures exactly 180°.

Example:



$\angle AOB$ has vertex O and the measure of $\angle AOB$ is 180°. In this case, we need to use three letters to name the angle.

When we name angles, if two or more angles share a ray and have a common vertex, then three letters are used to name each angle.



In the figure above:

The top angle ($\angle AOB$) can also be named $\angle 1$ or $\angle BOA$, but not $\angle O$.

$\angle 2$ can be named $\angle BOC$ or $\angle COB$.

$\angle 3$ can be named $\angle COD$ or $\angle DOC$.

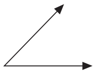
Also, $\angle AOC$ is composed of $\angle 1 + \angle 2$ and

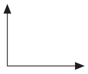
$\angle BOD$ is composed of $\angle 2 + \angle 3$ and

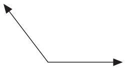
$\angle AOD$ is composed of $\angle 1 + \angle 2 + \angle 3$

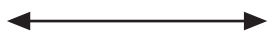
Practice

Match each **description** *with the correct term.*

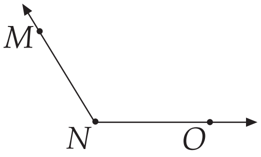
______ 1. < 90°  A. acute angle

______ 2. = 90°  B. obtuse angle

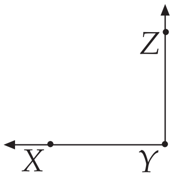
______ 3. > 90° and < 180°  C. right angle

______ 4. = 180°  D. straight angle

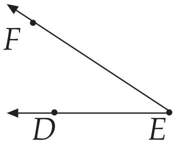
Compare each **angle** *below with a* **right angle***. Then write whether the angle is* **acute**, **obtuse**, *or* **right**.



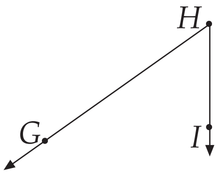
5. ______________ angle



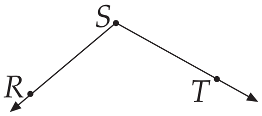
6. ______________ angle



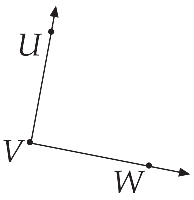
7. ______________ angle



8. ______________ angle



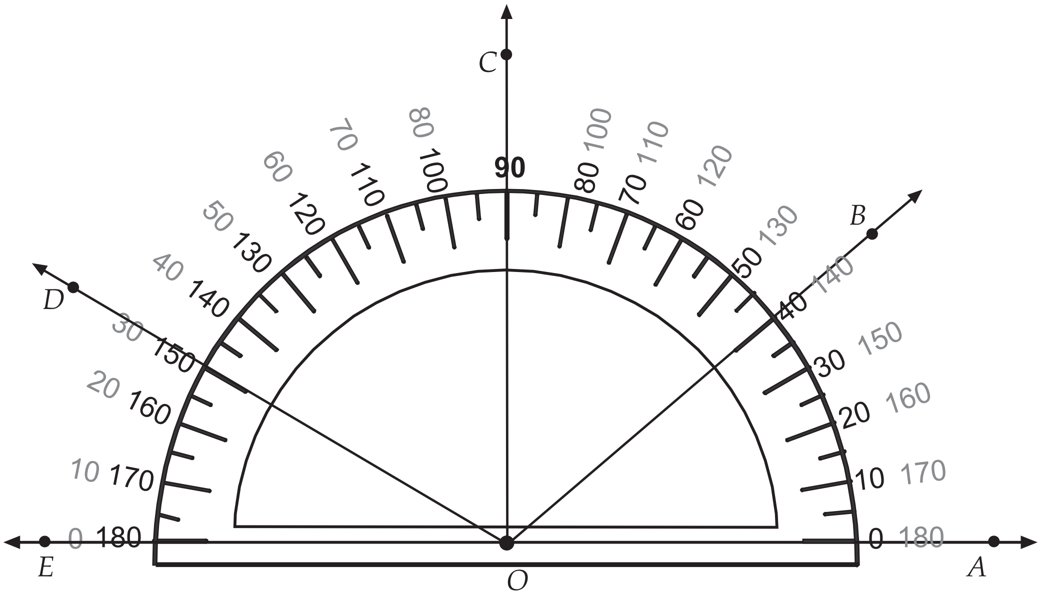
9. ______________ angle



10. ______________ angle

Practice

Use the **protractor** *below to find the* **measure of each angle***. Then write whether the angle is* **acute**, **right**, *or* **obtuse**.



1. ∠AOB ______________________________

2. ∠AOC ______________________________

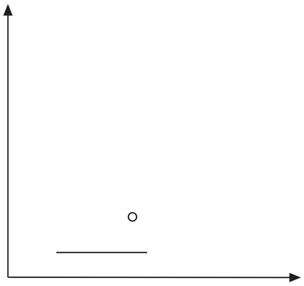
3. ∠BOD ______________________________

4. ∠COD ______________________________

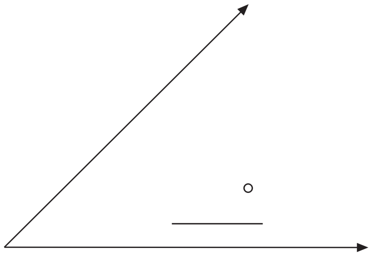
5. ∠COE ______________________________

6. ∠DOE ______________________________

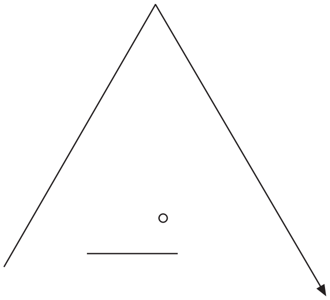
Use a **protractor** *to* **measure** *each* **angle** *below.*

7. 

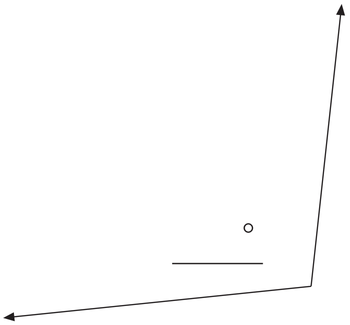
_______°

10. 

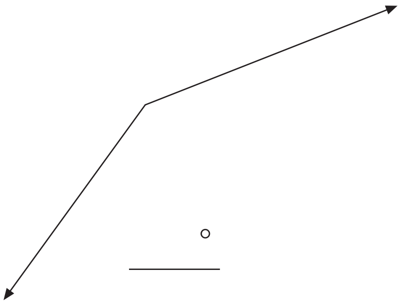
_______°

8. 

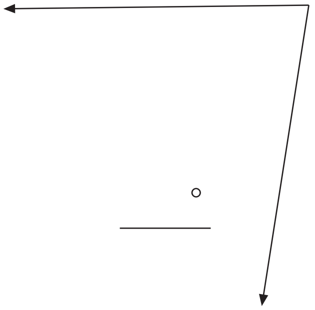
_______°

11. 

_______°

9. 

_______°

12. 

_______°

Use a **protractor** *to* **draw angles** *having these* **measures**.

13. 60°

17. 160°

14. 120°

18. 45°

15. 90°

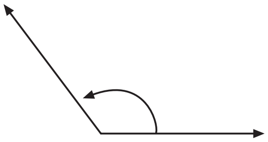
19. 100°

16. 20°

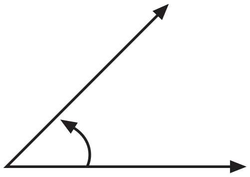
Practice

Match each **graphic** *with the correct term. Write the letter on the line provided.*

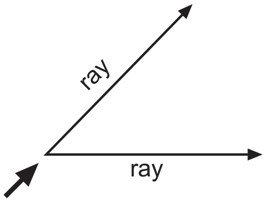
______ 1.



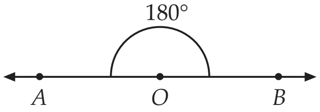
______ 2.



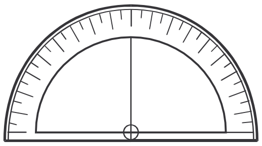
______ 3.



______ 4.



______ 5.



______ 6.

A. acute angle

B. obtuse angle

C. protractor

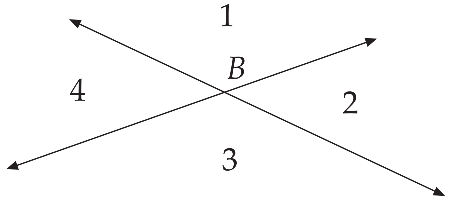
D. right angle

E. straight angle

F. vertex

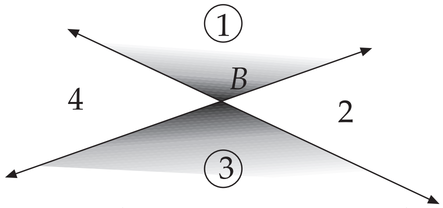
More Special Angles

Suppose we draw 2 lines that intersect:



Notice that 4 angles are formed. Which angles appear to be congruent? Test your theory by measuring each angle.

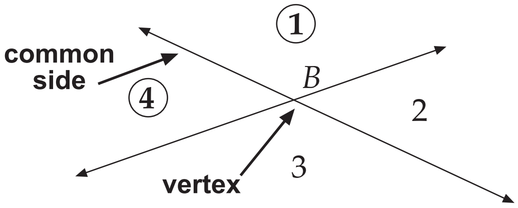
You should have found that angle 1 and angle 3 are congruent and angle 2 and angle 4 are congruent.



Angle 1 and angle 3 are congruent and angle 2 and angle 4 are congruent.

When two lines intersect, angles that are *opposite* or directly across from each other are called **vertical angles**. Vertical angles are always congruent.

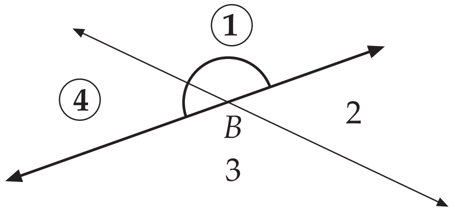
Angle 1 and angle 4 have a common side and also the same vertex. We say that they are **adjacent angles** because they are next to each other.



Angle 1 and angle 4 have a common side and also the same vertex. We say they are adjacent angles because they are next to each other.

Angle 1 and angle 2 are also *adjacent angles*, as well as angle 2 and angle 3, and angle 3 and angle 4.

Again look at angle 1 and angle 4. We see that the two angles together make a straight angle.



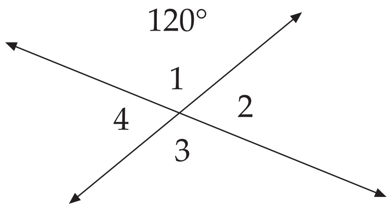
Angle 1 and angle 4 together make a straight angle.

Remember, a straight angle is an angle that measures 180 degrees. These two angles share a special relationship:

Two angles are said to be **supplementary angles** if the **sum** of their measures is 180 degrees. Therefore angle 1 and angle 4 are *supplementary angles*.

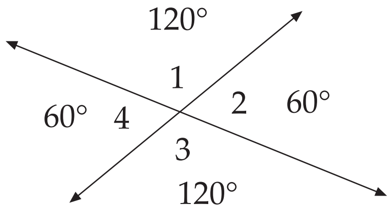
Example: Let's consider the following situation.

Suppose that the measure of angle 1 is 120 degrees. Can we find the measure of the other angles without measuring?



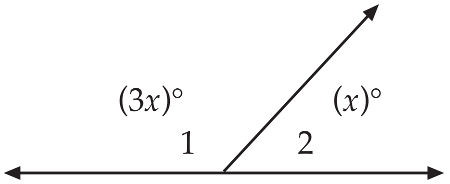
Remember that vertical angles (opposite angles) are congruent. Therefore the measure of angle 3 is also 120 degrees.

Angle 1 and angle 4 are supplementary angles (measure of angles add to equal 180 degrees). If angle 1 is 120 degrees, then angle 4 has to be 60 degrees.



Angle 2 and angle 4 are vertical angles, so if angle 4 is 60 degrees, then angle 2 is also 60 degrees.

Example: Study the following problem.



It appears that angle 1 and angle 2 are supplementary. If the angles are supplementary, their measures add to 180 degrees:

Remember: m∠1 means measure of angle 1

$$\begin{aligned} m\angle 1 + m\angle 2 &= 180^\circ \\ (3x)^\circ + (x)^\circ &= 180^\circ \qquad \text{combine like terms} \\ (4x)^\circ &= 180^\circ \\ \frac{4x^\circ}{4} &= \frac{180^\circ}{4} \qquad \text{divide both sides by 4} \\ x &= 45^\circ \end{aligned}$$

To check, **substitute** 45 for the **variable** *x* in the **equation**.

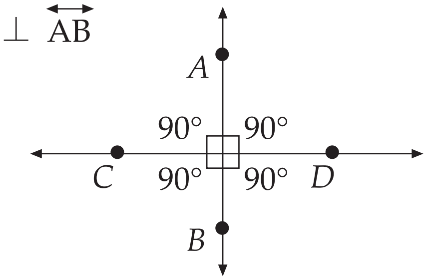
Measure of angle 1:	Measure of angle 2:
$3x = 3(45)$	$x = 45$ degrees
$= 135$ degrees	

Check: 135 degrees + 45 degrees = 180 degrees *It checks!*

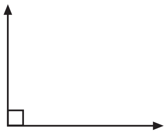
What if we have 2 intersecting lines and all 4 of the angles formed measure 90 degrees? In this situation, we say that the lines are **perpendicular lines**. They form right angles. Two lines (or rays, or segments) that meet at right angles are **perpendicular (⊥)**. *Perpendicular lines* form right angles where they meet.

Note: The symbol ⊥ means *is perpendicular to*.

$\overleftrightarrow{AB} \perp \overleftrightarrow{AB}$



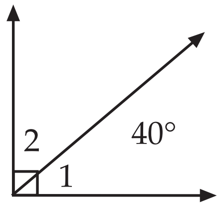
Remember: If you see an angle with a little box "inside it," then you know that the measure of the angle is 90 degrees and that the angle is a right angle.



Our discussion of right angles and perpendicular lines leads to the last definition in our section.

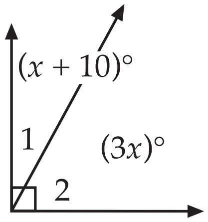
Two angles are **complementary angles** if the sum of their measures is 90 degrees.

Example: Study the following picture.



Angle 1 and angle 2 are *complementary angles* because the sum of their measures is 90 degrees. (Note the little box in the corner). If angle 1 measures 40 degrees, then angle 2 would have to be 50 degrees, because 40 + 50 = 90.

Example: Study the following picture.



This problem is the same as the one before it, only a little bit harder.

Since we see a little box in the corner, we know that the two angles are complementary. The sum of their measures is 90 degrees.

$$\begin{aligned} m\angle 1 + m\angle 2 &= 90^\circ \\ (x + 10)^\circ + (3x)^\circ &= 90^\circ \\ (4x) + (10^\circ - 10^\circ) &= 90^\circ - 10^\circ && \text{combine like terms and} \\ 4x &= 80^\circ && \text{subtract 10 from both sides} \\ \frac{4x}{4} &= \frac{80}{4} && \text{divide both sides by 4} \\ x &= 20^\circ \end{aligned}$$

To check, *substitute* 20 for the variable x.

Measure of angle 1:

$$\begin{aligned} x + 10 &= 20 + 10 \\ &= 30 \text{ degrees} \end{aligned}$$

Measure of angle 2:

$$\begin{aligned} 3x &= 3(20) \\ &= 60 \text{ degrees} \end{aligned}$$

Check: 30 degrees + 60 degrees = 90 degrees *It checks!*

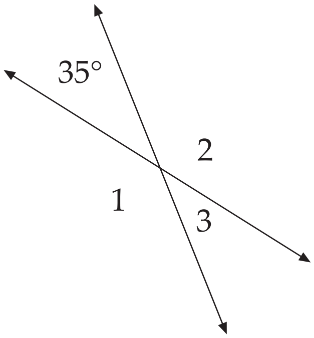
Practice

Find the **measures of these angles**. *Show all your work.*

1. m∠1 = ____________

   m∠2 = ____________

   m∠3 = ____________

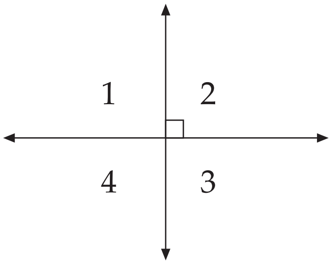


2. m∠1 = ____________

   m∠2 = ____________

   m∠3 = ____________

   m∠4 = ____________



Answer the following. Show all your work.

3. ∠*A* and ∠*B* are complementary. If the m∠*A* is 28 degrees, what is the m∠*B*? ____________

4. ∠*A* and ∠*B* are supplementary. If the m∠*A* is 28 degrees, what is the m∠*B*? ____________

5. Determine if the angles are complementary *or* supplementary.

   a. m$\angle A$ = 15 degrees and m$\angle B$ = 75 degrees

      The angles are ____ .

   b. m$\angle C$ = 100 degrees and m$\angle D$ = 80 degrees

      The angles are ____ .

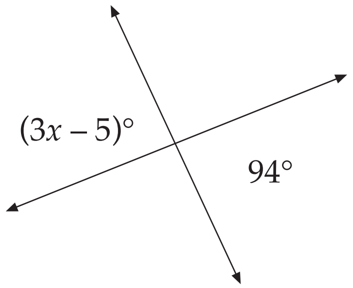
   c. m$\angle E$ = 90 degrees and m$\angle F$ is a right angle

      The angles are ____ .

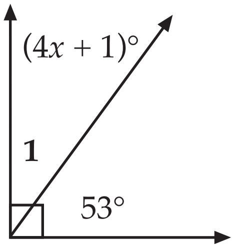
   d. m$\angle G$ = $(2x - 90)$ degrees and m$\angle H$ = $(180 - 2x)$ degrees

      The angles are ____ .

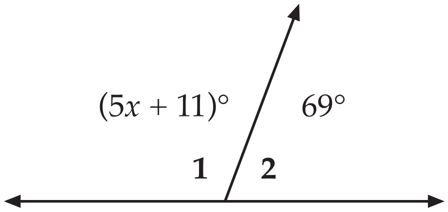
6. Solve for x. ____________



7. Solve for x. Find the measure of angle 1. ______________________



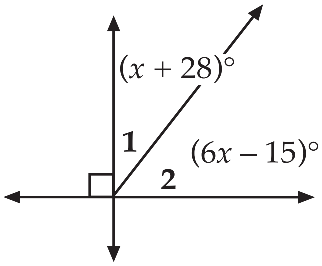
8. Solve for x. Find the measure of angle 1. ______________________



Find the **measures of these angles**. *Show all your work.*

9. m∠1 = ______________

m∠2 = ______________

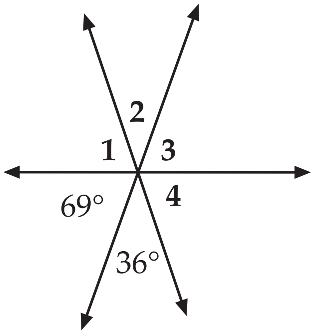


10. m∠1 = ______________

m∠2 = ______________

m∠3 = ______________

m∠4 = ______________

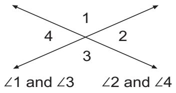


Practice

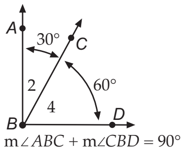
Use the list below to write the correct term for each definition on the line provided.

adjacent angles
complementary angles
degree (°)
measure (m) of an angle (∠)
perpendicular lines
substitute
sum
supplementary angles
variable
vertical angles

____________________ 1. the opposite angles formed when two lines intersect

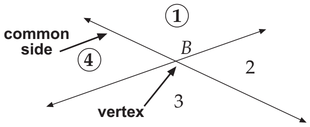


____________________ 2. two angles, the sum of which is exactly 90°

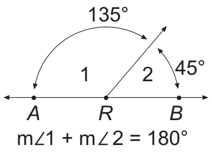


____________________ 3. the result of an addition

____________________ 4. two angles having a common vertex and sharing a common side



____________________ 5. two angles, the sum of which is exactly 180°



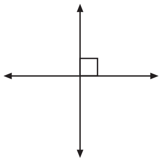
____________________ 6. the number of degrees (°) of an angle

____________________ 7. common unit used in measuring angles

____________________ 8. to replace a variable with a numeral

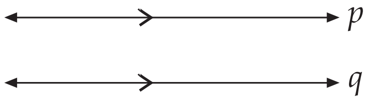
____________________ 9. any symbol that could represent a number

____________________ 10. two lines that intersect to form right angles



Exploring Parallel Lines

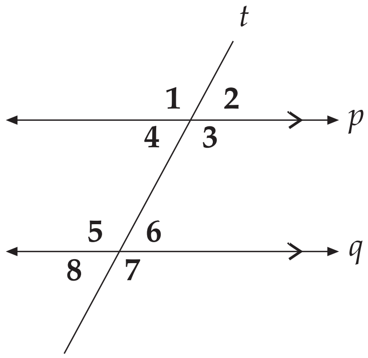
Two (or more) lines in a plane that do *not* intersect are said to be **parallel (||)**. **Parallel lines** remain the same distance apart. They will *never* intersect, even if extended.



Lines can be labeled using a lower case letter.
The bold arrows (>) on the lines indicate that the lines are parallel.

Line *p* is *parallel* to line *q*. The symbol (||) means *is parallel to*. Therefore, we can use the following notation: *p* || *q*

Suppose we draw a third line *t* which intersects the two *parallel lines* (||) *p* and *q*. The third line *t* is called a **transversal**. A *transversal* is a line that intersects two or more (usually parallel) lines.



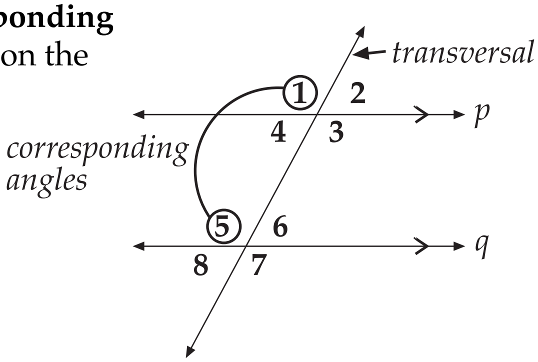
Notice that 8 angles are formed. Look at the 4 angles at the top.

$\angle 1$ is in the upper left
$\angle 2$ is in the upper right
$\angle 3$ is in the lower right
$\angle 4$ is in the lower left

Now look at the bottom 4.

$\angle 5$ is in the upper left, just like $\angle 1$
$\angle 6$ is in the upper right, just like $\angle 2$
$\angle 7$ is in the lower right, just like $\angle 3$
$\angle 8$ is in the lower left, just like $\angle 4$

When parallel lines are cut by a transversal, angles in the same relative or matching position are called **corresponding angles**. *Corresponding angles* also lie on the same side of a transversal.



Therefore

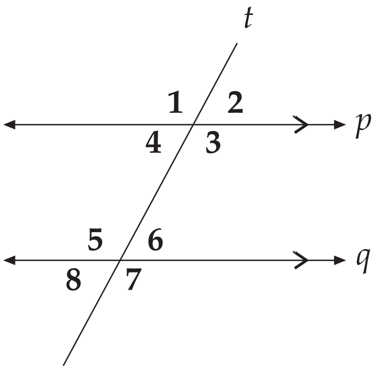
$\angle 1$ corresponds to $\angle 5$;
$\angle 6$ corresponds to $\angle 2$;
$\angle 7$ corresponds to $\angle 3$;
and $\angle 8$ corresponds to $\angle 4$.

Measure each pair of angles with a protractor. Label each angle with its measure.

Did you find the following?

> If you have parallel lines cut by a transversal, then the measures of the corresponding angles are equal.

Look at the angles again.



Since $\angle 1$ and $\angle 5$ are corresponding angles, we know that

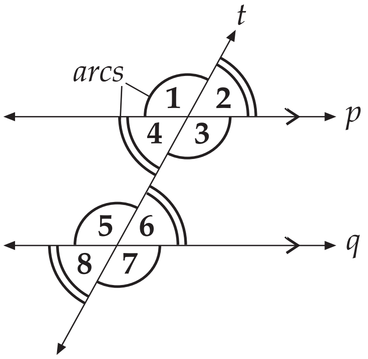
$m\angle 1 = m\angle 5$

We also know that $\angle 1$ and $\angle 3$ are vertical angles, and $\angle 5$ and $\angle 7$ are vertical angles. Remember that the measures of vertical angles are equal. Therefore,

$m\angle 1 = m\angle 3$
$m\angle 5 = m\angle 7$

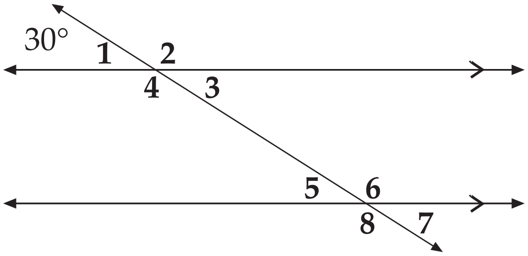
Logically, we can say that $m\angle 1 = m\angle 3 = m\angle 5 = m\angle 7$.
Similar logic would allow us to say that $m\angle 2 = m\angle 4 = m\angle 6 = m\angle 8$.

To summarize what has been said we will mark all angles having the same measure with *arcs*. Use the arcs marked on the angles as your guide to which angles have the same measure.



What is the relationship between ∠1 and ∠2? If you said that the angles are supplementary, then you are correct! The two angles together make a straight angle. A straight angle is an angle that measures 180 degrees.

Now see if you can unravel the following mystery:



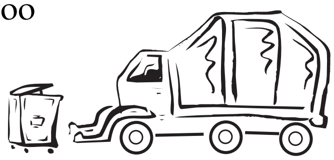
Find the measures of all angles if you know that m∠1 = 30°.

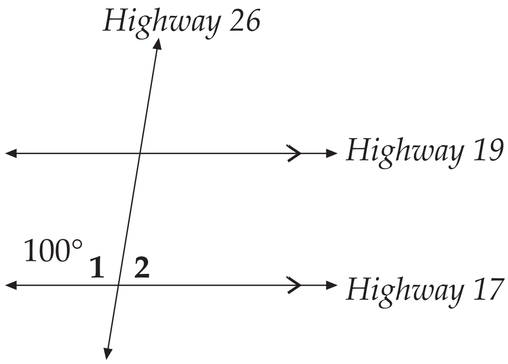
Solution: If m∠1 = 30°, then m∠2 = 150°.
∠1 and ∠2 are supplementary, and their measures add to 180°.

m∠1 = m∠3 = m∠5 = m∠7 = 30°
m∠2 = m∠4 = m∠6 = m∠8 = 150°

Here is another problem to consider:

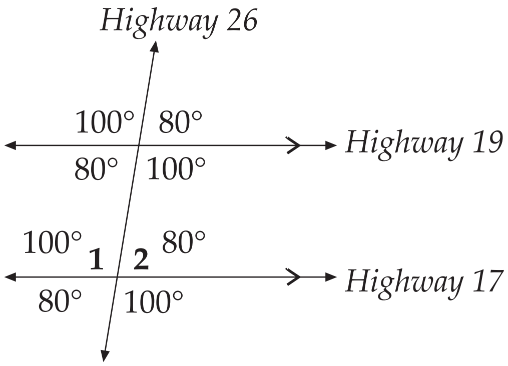
The city of Big Hill, Florida is considering expanding garbage service to a nearby community. There is one problem. The city has to be sure that the angles at the intersections will not be too sharp for the garbage trucks to make the turns. A garbage truck cannot turn at an angle that is less than 70°. Can the garbage service be instituted if we know that one of the corners measures 100°?





Solution: Label all angles whose measures equal 100 degrees. (corresponding and vertical). Remember, ∠1 and ∠2 are supplementary, so m∠2 must be 80°.

Now label the corresponding vertical and corresponding angles. You see that there are no corners where an angle measures less than 70°, so the garbage service can be extended.

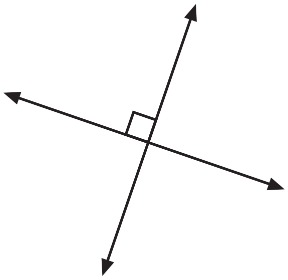


Practice

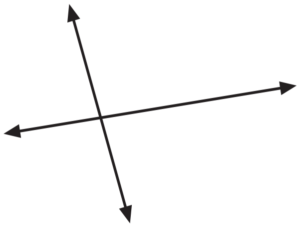
Write || *if the lines are* **parallel**. *Write* ⊥ *if the lines are* **perpendicular**. *If the lines are* ***neither*** *parallel* ***nor*** *perpendicular, write* **neither**.

1. ______________

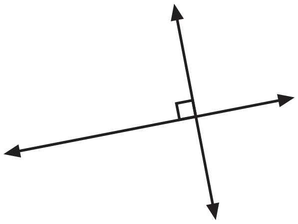
2. ______________



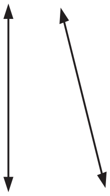
3. ______________



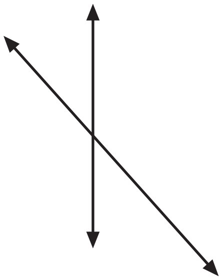
4. ______________



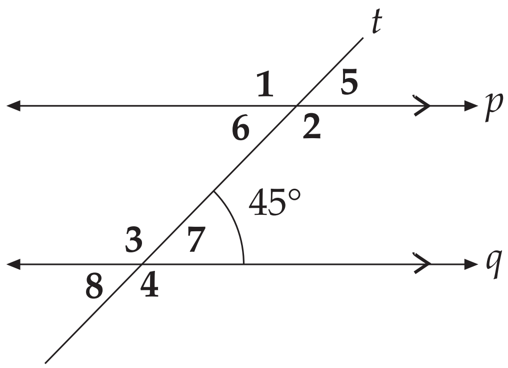
5. ______________



6. ______________



Use the figure below to answer the following.



7. Which two lines are parallel? ______________________________

8. Name four pairs of corresponding angles. ___________________

____

9. Name four pairs of vertical angles.__________________________

____

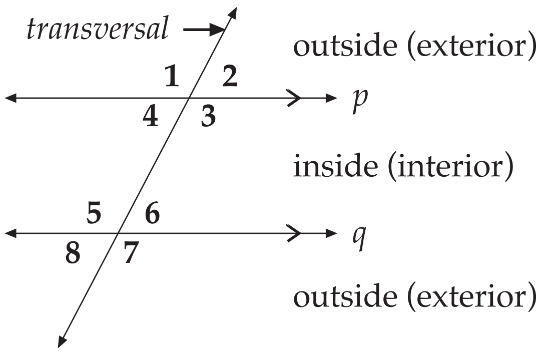
10. Find the m∠1. ____

11. Find the m∠2. ____

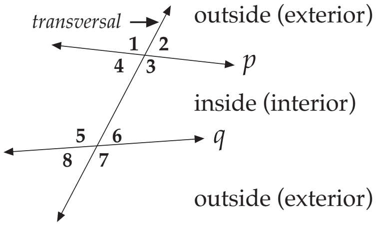
12. Find the m∠6. ____

Angles Formed by a Transversal

When a transversal intersects two lines, it forms eight angles. These angles are **alternative angles**. *Alternative angles* lie on opposite sides and at opposite ends of a transversal. The four angles lying between or *inside* the two lines are *alternate interior angles*. The four angles lying *outside* the two lines are *alternate exterior angles*.



Note: Even when lines cut by a transversal are *not* parallel, we still use the same vocabulary.



However, there are special properties when the lines intersected by a transversal are parallel.

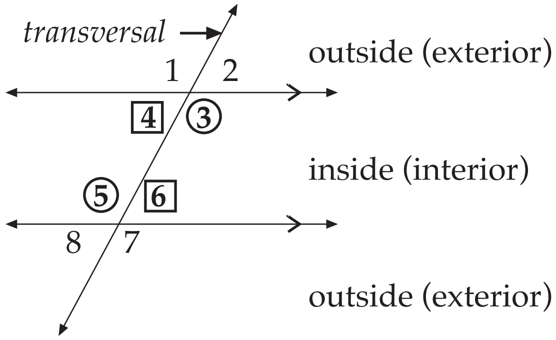
Remember that we have shown that when a transversal intersects two parallel lines:

$m\angle 1 = m\angle 3 = m\angle 5 = m\angle 7$ *and*
$m\angle 2 = m\angle 4 = m\angle 6 = m\angle 8$

Alternate Interior and Alternate Exterior Angles

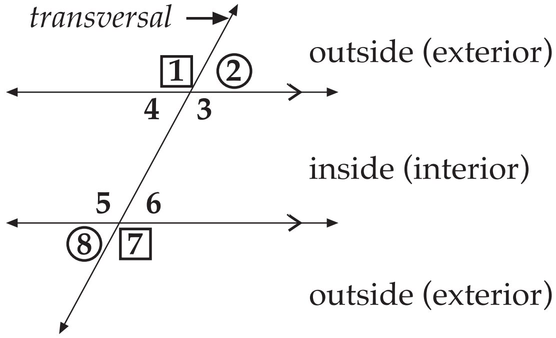
Angles 3, 4, 5, and 6 are called *alternate interior angles* because they are on the *inside* of the parallel lines. Angles 1, 2, 7, and 8 are called *alternate exterior angles* because they are on the *outside* of the parallel lines.

∠3 and ∠5
and
∠4 and ∠6 are alternate interior angles (opposite inside).



Alternate exterior angles are on *opposite sides* of the transversal and on the *outside* of the lines it intersects. There are 2 pairs of alternate exterior angles.

∠1 and ∠7
and
∠2 and ∠8 are alternate exterior angles (opposite outside).

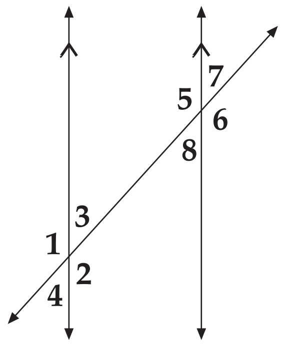


If we have parallel lines cut by a transversal, then the following is true:

- the measures of alternate interior angles are equal
- the measures of alternate exterior angles are equal.

Practice

Use the figure below to answer the following.



1. Name two pairs of alternate interior angles. ____________________

   ____

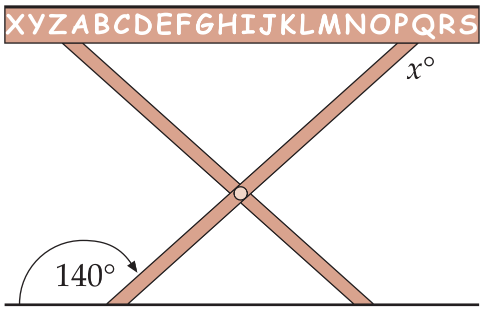
2. Name two pairs of alternate exterior angles. ____________________

   ____

3. If angle 4 measures 30°, what is the measure of angles 6 and 7? __

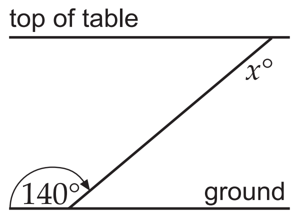
   ____

4. Max is building a child's play table. Here is an end view of the table.



The top of the table will be parallel to the ground. What is the measure of angle x?

Hint: Redraw the picture.

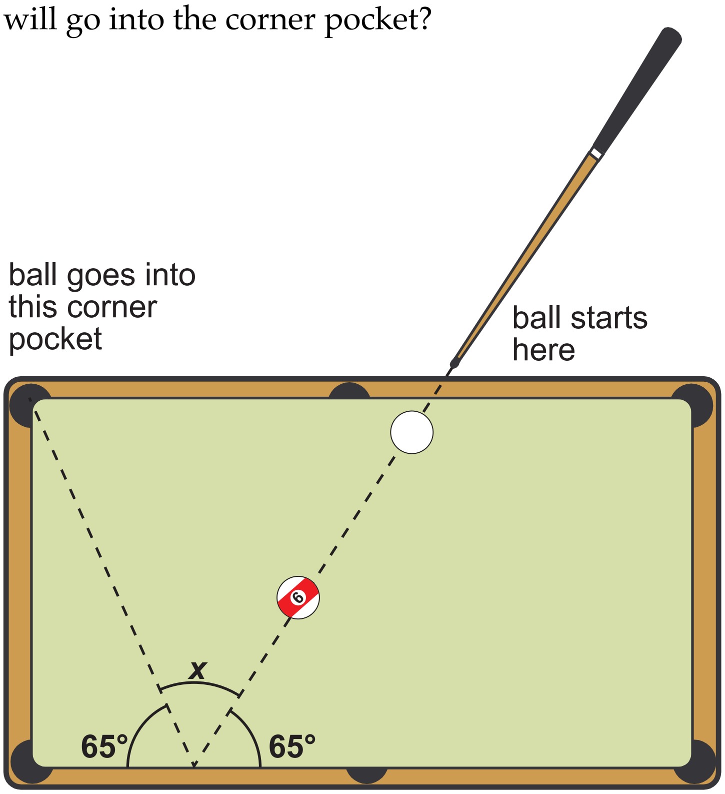


a. 140º
b. 40º
c. 180º
d. 90º

Number 5 is a **gridded-response item***.*

Write answer along the top of the grid and correctly mark it below.

5. Chip is playing pool. He wants to hit the ball into the corner pocket as shown below. What angle (x) must the path of the ball take so it will go into the corner pocket?



Mark your answer on the grid to the right.

	⊘	⊘	⊘	
⊙	⊙	⊙	⊙	⊙
⓪	⓪	⓪	⓪	⓪
①	①	①	①	①
②	②	②	②	②
③	③	③	③	③
④	④	④	④	④
⑤	⑤	⑤	⑤	⑤
⑥	⑥	⑥	⑥	⑥
⑦	⑦	⑦	⑦	⑦
⑧	⑧	⑧	⑧	⑧
⑨	⑨	⑨	⑨	⑨

Practice

Use the list below to complete the following statements.

alternate angles
corresponding angles
alternate exterior angles
alternate interior angles
parallel
parallel lines
transversal

1. ____________________ are always the same distance apart and will never intersect, even if extended.

2. A ____________________ is a line that intersects two or more (usually parallel) lines.

3. When parallel lines are cut by a *transversal*, angles in the same relative or matching position that lie on the same side of a transversal are called ____________________ .

4. When a transversal intersects two lines, it forms eight angles. The four angles lying between or *inside* the two lines are ____________________ .

5. Two (or more) lines in a plane that do *not* intersect are said to be ____________________ .

6. When a transversal intersects two lines, it forms eight angles. The four angles lying *outside* the two lines are ____________________ .

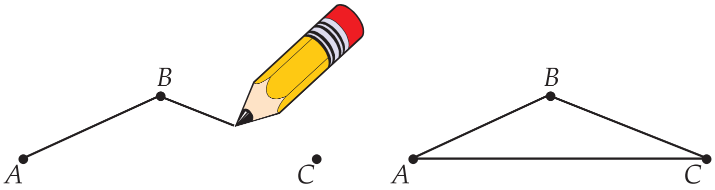
7. A pair of angles that lie on the *opposite side* and at opposite ends of a transversal are called ____________________ .

Lesson Two Purpose

- Understand concrete and symbolic representations of real numbers in real-world situations. (MA.A.1.4.3)
- Add, subtract, multiply, and divide real numbers, including square roots and exponents, using appropriate methods of computing, such as mental mathematics, paper and pencil, and calculator. (MA.A.3.4.3)
- Use estimation strategies in complex situations to predict results and to check the reasonableness of results. (MA.A.4.4.1)
- Relate the concepts of measurement to similarity and proportionality in real-world situations. (MA.B.1.4.3)
- Represent and apply geometric properties and relationships to solve real-world and mathematical problems including ratio and proportion. (MA.C.3.4.1)
- Using a rectangular coordinate system (graph), apply and algebraically verify properties of two-dimensional figures. (MA.C.3.4.2)

Triangles

Draw three points on a piece of paper. Make sure they do not lie on the same line. Label one point *A*, one point *B*, and one point *C*. Connect the points. The figure formed is, of course, a **triangle**. In a *plane* (a flat surface), a closed figure formed by three line segments is a *triangle*.

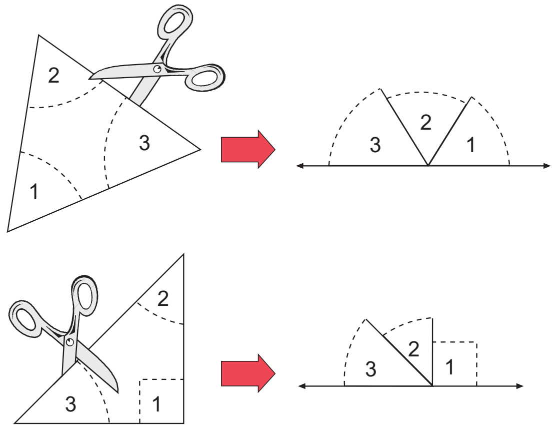


The points, A, B, and C are called the *vertices of the triangle*. If we want to talk about just one point, we would use the word *vertex*, the singular for *vertices*. The triangle we drew is triangle *ABC*. Vertices are named in a clockwise or counterclockwise manner.

Note: $\triangle ABC$ means triangle *ABC*.

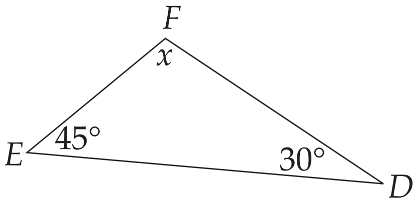
Angles of a Triangle

Let's investigate the three angles of a triangle. Take a piece of paper and draw a large triangle. Put numbers in each angle. Cut the triangle out. Now cut the angles off as shown below. Fit the angle pieces with the points together as shown below. What do the three angles make? You should have found that the angles make a straight angle.



This demonstrates that the sum of the measures of the three angles in *any* triangle is 180 degrees.

Example:



What is the m∠*F*?

Solution:

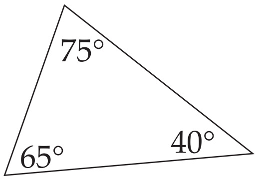
$$\begin{array}{llllll} m\angle D & + & m\angle E & + & m\angle F & = 180^\circ \\ 30^\circ & + & 45^\circ & + & x^\circ & = 180^\circ \\ & 75^\circ & & + & x^\circ & = 180^\circ \\ 75^\circ & - & 75^\circ & + & x^\circ & = 180^\circ - 75^\circ \quad \text{subtract 75 from both sides} \\ & & & & x^\circ & = 105^\circ \end{array}$$

Classifying Triangles

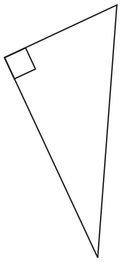
Triangles are **polygons** with three sides. *Polygons* are closed plane figures whose sides are straight and do not cross. Triangles are classified in two different ways. Triangles are classified either by the measure of their angles or the measure of their sides. However, no matter how a triangle is classified, the sum of the measures of the angles in a triangle is 180 degrees.

Triangles Classified by Their Angles—Acute, Right, Obtuse, and Equiangular

An **acute triangle** contains *all* acute angles with measures less than 90°.

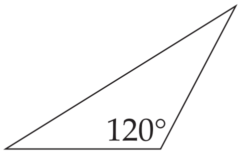


A **right triangle** contains *one* right angle with a measure of 90°.

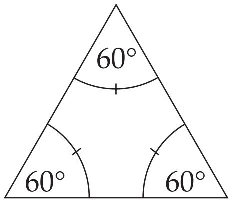


Note: right triangles are marked ∟

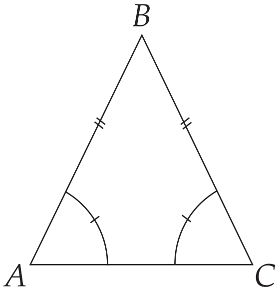
An **obtuse triangle** contains *one* obtuse angle with a measure of more than 90° but less than 180°.



An **equiangular triangle** contains *all* equal angles, each with a measure of 60°.



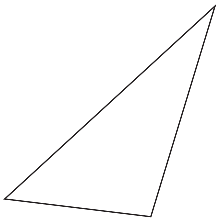
Note: *Tick marks* are used to denote *angles* or *sides* with the same measure. *Arcs* are also used to show *angles* with the same measure.



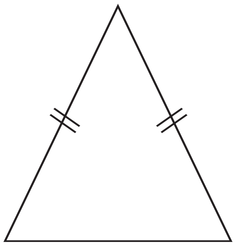
$$m\angle A = m\angle C$$
$$AB = BC$$

Triangles Classified by the Lengths of Their Sides—Scalene, Isosceles, and Equilateral

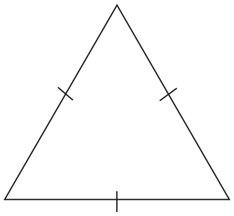
A **scalene triangle** has *no* congruent sides—no sides are the same length.



An **isosceles triangle** has at least *two* congruent sides—two or more sides are the same length.

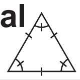
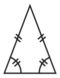
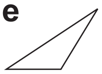


An **equilateral triangle** has *three* congruent sides—*all* sides are the same length.



So, all triangles may be classified by their angles (acute, right, obtuse, or equiangular), by their sides (equilateral, isosceles, scalene), or both. See the chart below.

Triangles

Classification	**Acute** < 90°	**Right** = 90°	**Obtuse** > 90° and <180°	**Equiangular** all = 60°
Equilateral 	✓			✓
Isosceles 	✓	✓	✓	✓
Scalene 	✓	✓	✓	

< means less than

> means greater than

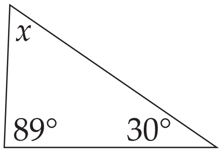
As you see in the above chart, a right triangle *may be* either isosceles or scalene, but is *never* acute, equilateral, or obtuse.

Practice

Find the **value of** x*. Then classify each triangle as* **acute**, **right**, **obtuse**, *or* **equiangular**.

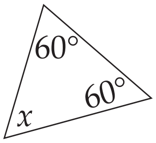
value of x **classification of triangle**

1. $x =$ __________



____

2. $x =$ __________



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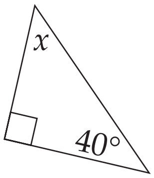
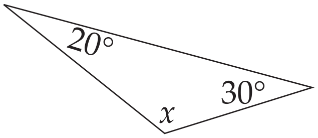
3. $x =$ __________



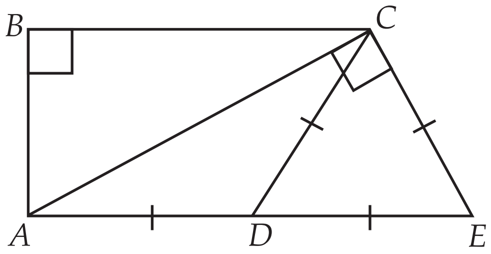
____

4. $x =$ __________



____

Use the figure below to answer the following.



5. Name all the right triangle(s). ______________________________

6. Name the isosceles triangle(s). ______________________________

7. Name the obtuse triangle(s). ____________________

8. Name the equilateral triangle(s). ____________________

Draw the following.

9. Draw an isosceles right triangle.

10. Draw an obtuse scalene triangle.

11. Draw a right scalene triangle.

Answer the following.

12. Can a triangle have two right angles? ____________________

Why or why not? ____________________

____

____

13. Can a triangle have two obtuse angles? ____________________

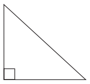
Why or why not? ____________________

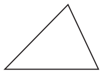
____

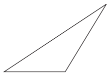
____

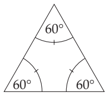
Practice

Match each **figure** *with the correct term.*

______ 1.  has one 90° angle

______ 2.  all angles are less than 90°

______ 3.  has one angle more than 90° but less than 180°

______ 4. 

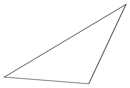
all angles are equal

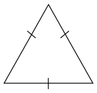
A. acute triangle

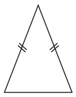
B. equiangular triangle

C. obtuse triangle

D. right triangle

______ 5.  has no congruent sides

______ 6.  has 3 congruent sides

______ 7.  has 2 congruent sides

A. equilateral triangle

B. isosceles triangle

C. scalene triangle

Practice

Use the list below to write the correct term for each definition on the line provided.

acute triangle	**isosceles triangle**	**right triangle**
equiangular triangle	**obtuse triangle**	**scalene triangle**
equilateral triangle	**polygon**	**triangle**

________________________ 1. a polygon with three sides; the sum of the measures of the angles is 180°

________________________ 2. a triangle with one right angle

________________________ 3. a triangle with one obtuse angle

________________________ 4. a triangle with three equal angles

________________________ 5. a triangle with three acute angles

________________________ 6. a triangle with three congruent sides

________________________ 7. a triangle with at least two congruent sides and two congruent angles

________________________ 8. a triangle with no congruent sides

________________________ 9. a closed plane figure whose sides are straight and do not cross

Similar Figures

Two figures that have the same shape are called **similar figures**. We will consider similar triangles as well as other similar shapes.

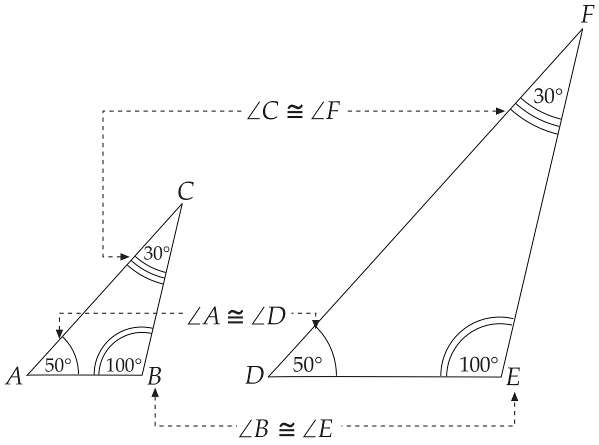
Similar figures are alike in a very specific way. All of the following must be true for two figures to be called similar figures.

- They have the same shape.
- The **corresponding angles** are congruent (have the same measure).
- The **ratios** (comparison of two quantities) of the lengths of the **corresponding sides** are equal—the sides are proportional in length.
- They may or may not have the same size or be in the same position.

We will use this knowledge and our skills in setting up and solving **proportions**—mathematical sentences stating two ratios are equal.

Look at these two triangles.

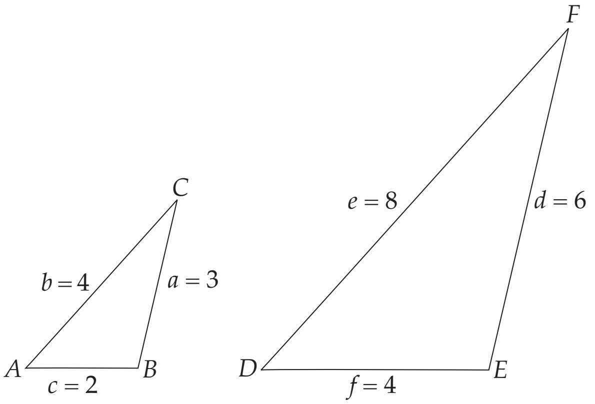
Remember: The symbol ≅ means is *congruent to*.



All of the following corresponding angles are equal.

$\angle A$ corresponds to $\angle D$
$\angle B$ corresponds to $\angle E$
$\angle C$ corresponds to $\angle F$

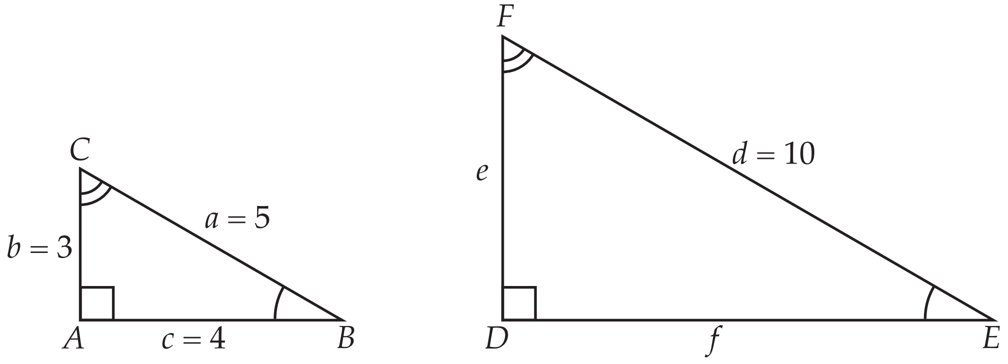
and



All of the following pairs of corresponding sides are *proportional* and have the same ratio.

$$\begin{array}{ccccccc} \frac{a}{d} & = & \frac{b}{e} & = & \frac{c}{f} & & \\ \downarrow & & \downarrow & & \downarrow & & \\ \frac{3}{6} & = & \frac{4}{8} & = & \frac{2}{4} & = & \frac{1}{2} \end{array}$$

These triangles are similar.



The symbol ~ means *is similar to*.

Therefore $\triangle ABC \sim \triangle \mathrm{DEF}$.

The single and double arcs indicate corresponding pairs of angles. ∠A corresponds to ∠*D*, ∠*B* corresponds to ∠*E*, and ∠*C* corresponds to ∠F.

$$\angle \mathrm{A} \cong \angle D; \angle B \cong \angle E; \angle C \cong \angle \mathrm{F}$$

We can write: $\frac{a}{d} = \frac{b}{e} = \frac{c}{f}$ and replace known values to find length of the sides.

$$\frac{5}{10} = \frac{3}{e} = \frac{4}{f}$$

The $\frac{5}{10}$ is our **scale factor** and is used to set up proportions for finding the values of *e* and *f*. The *scale factor* is the ratio between the lengths of corresponding sides of two similar figures. And you know that pairs of corresponding sides in similar figures have equal ratios.

You can solve a proportion by using **cross products**. A *cross product* is the product of one *numerator* and the opposite *denominator* in a pair of fractions. The cross products of equivalent fractions will be equal.

$$\begin{array}{lcl} \frac{5}{10} = \frac{3}{e} & \text{and} & \frac{5}{10} = \frac{4}{f} \\ 5e = 30 & & 5f = 40 \\ \frac{5e}{5} = \frac{30}{5} & \longleftarrow \boxed{\text{divide both sides by 5}} \longrightarrow & \frac{5f}{5} = \frac{40}{5} \\ e = 6 & & f = 8 \end{array}$$

When we substitute our values for e and f into the ratios above, we get

$$\frac{a}{d} = \frac{b}{e} = \frac{c}{f}$$

$$\frac{5}{10} = \frac{3}{6} = \frac{4}{8}$$

Notice that each ratio reduces to $\frac{1}{2}$.

$$\frac{5}{10} = \frac{1}{2}$$

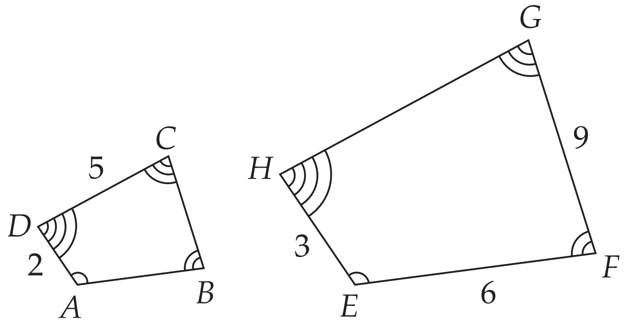
$$\frac{3}{6} = \frac{1}{2}$$

$$\frac{4}{8} = \frac{1}{2}$$

Another way to describe our result is to say:

5 is to 10 as 3 is to 6 as 4 is to 8.

The properties of similar triangles also apply to similar **quadrilaterals** (polygons with four sides).



$$\begin{aligned} m\angle A &= m\angle E \\ m\angle B &= m\angle F \\ m\angle C &= m\angle G \\ m\angle D &= m\angle H \end{aligned}$$

Also: $\frac{AB}{EF} = \frac{BC}{FG} = \frac{CD}{GH} = \frac{DA}{HE}$

Filling in the given side lengths gives us:

$$\frac{AB}{6} = \frac{BC}{9} = \frac{5}{GH} = \frac{2}{3}$$

From these ratios we see a scale factor of $\frac{2}{3}$, leading us to set up the following proportions.

$$\begin{array}{ccccc} \frac{AB}{6} = \frac{2}{3} & \text{and} & \frac{BC}{9} = \frac{2}{3} & \text{and} & \frac{5}{HG} = \frac{2}{3} \\ 3(AB) = 2 \bullet 6 & & 3(BC) = 2 \bullet 9 & & 5 \bullet 3 = 2(HG) \\ 3(AB) = 12 & & 3(BC) = 18 & & 15 = 2(HG) \\ \frac{3(AB)}{3} = \frac{12}{3} & & \frac{3(BC)}{3} = \frac{18}{3} & & \frac{15}{2} = \frac{2(HG)}{2} \\ AB = 4 & & BC = 6 & & 7\frac{1}{2} = HG \end{array}$$

Now substitute the values for *AB*, *BC*, and *HG* into the ratios above.

$$\begin{aligned} \frac{AB}{EF} &= \frac{BC}{FG} = \frac{CD}{GH} = \frac{DA}{HE} \\ \frac{4}{6} &= \frac{6}{9} = \frac{5}{7\frac{1}{2}} = \frac{2}{3} \end{aligned}$$

Notice that each ratio reduces to $\frac{2}{3}$.

$$\frac{4}{6} = \frac{2}{3}$$
$$\frac{6}{9} = \frac{2}{3}$$
$$\frac{5}{7\frac{1}{2}} = \frac{2}{3}$$

Another way to describe the results is to say: 4 is to 6 as 6 is to 9 as 5 is to $7\frac{1}{2}$.

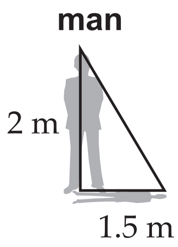
Indirect Measurement

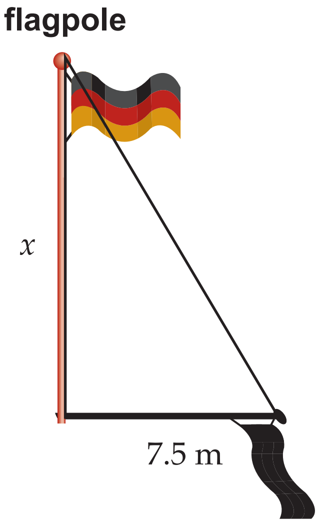
If you cannot measure a length directly, you can sometimes use similar triangles to make an indirect measurement.

Example:

Consider shadows made by a flagpole and a man. The man is 2 meters tall and casts a shadow 1.5 meters. The flagpole casts a shadow 7.5 meters in length. Let's find the height of the flagpole.

The right triangles formed by the flagpole and the man are similar triangles.





1. Identify two corresponding ratios.
   ratio of shadows: 1.5 to 7.5
   ratio of heights: 2 to x

2. Set up a proportion.

$$\begin{array}{r} \text{shadow of man} \rightarrow \\ \text{shadow of flagpole} \rightarrow \end{array} \frac{1.5}{7.5} = \frac{2}{h} \begin{array}{l} \leftarrow \text{height of man} \\ \leftarrow \text{height of flagpole} \end{array}$$

3. Solve the proportions.

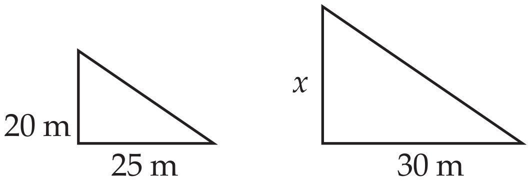
$$\begin{array}{rcll} \frac{1.5}{7.5} & = & \frac{2}{h} & \\ 1.5 \text{ x } h & = & 7.5 \text{ x } 2 & \text{find cross products} \\ 1.5 \text{ x } h & = & 15 & \text{solve for } h \\ \frac{1.5h}{1.5} & = & \frac{15}{1.5} & \text{divide each side by 1.5} \\ h & = & 10 & \text{answer: height of the flagpole is 10 meters} \end{array}$$

We can see that the *scale factor* is 5 because the shadow of the flagpole is 5 times as long as the shadow of the man. The flagpole will be 5 times as tall as the man or 5 times 2 or 10 meters.

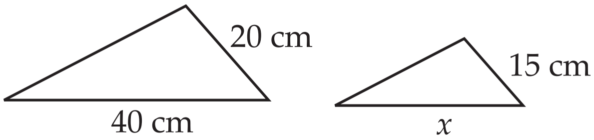
Practice

For each problem below, the two figures are **similar**. *Set up and solve a* **proportion** *to find the* **length** ***x***. *Show all your work.*

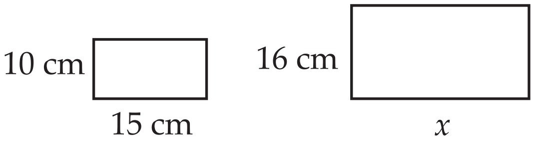
1.



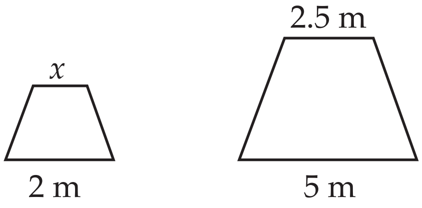
2.



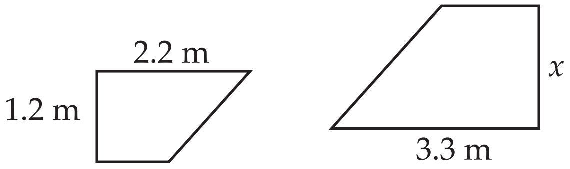
3.



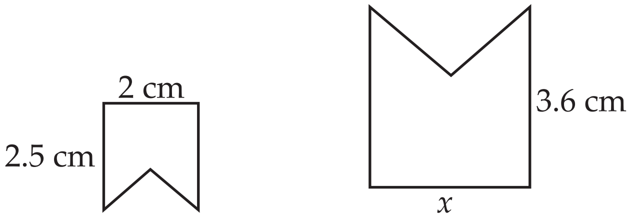
4.



5.

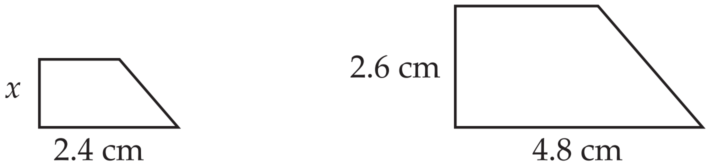


6.



7.

8.



Draw a **diagram**, *set up a* **proportion**, *and* **solve** *the following.*

9. When a high-rise office building casts a 70-foot shadow, a man 6 feet tall casts a 3-foot shadow. How tall is the building?

10. A TV tower is 35 meters high and casts a 10-meter shadow. How tall is a nearby tree that casts a 6-meter shadow at the same time?

Practice

Match each definition with the correct term. Write the letter on the line provided.

______ 1. the quotient of two numbers used to compare two quantities

______ 2. the product of one numerator and the opposite denominator in a pair of fractions

______ 3. polygon with four sides

______ 4. the matching angles and sides in similar figures

______ 5. figures that have the same shape but not necessarily the same size

______ 6. a mathematical sentence stating that two ratios are equal

______ 7. the ratio between the lengths of corresponding sides of two similar figures

A. corresponding angles and sides

B. cross product

C. proportion

D. quadrilateral

E. ratio

F. scale factor

G. similar figures

The Coordinate System

Before we can continue our study of geometry, we need to know how to **graph points** on a plane. You probably have studied graphing points before. Below is a model of a **coordinate grid or system**. A *coordinate grid or system* is a two-dimensional system used to locate points in a plane. Study the picture and see how much you remember!

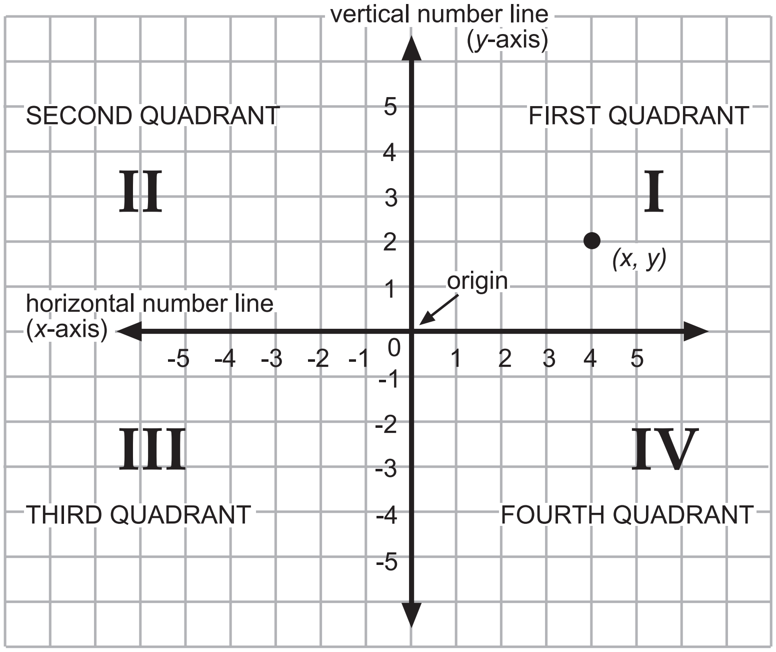
1. The coordinate system has a horizontal (⟷) number line called the ***x*-axis** and a vertical (↕) number line called the ***y*-axis**.

2. The two number lines or **axes of a graph** are perpendicular and *intersect* (meet) at a point called the **origin**. The **coordinates** at the **intersection** of the *origin* are (0, 0).

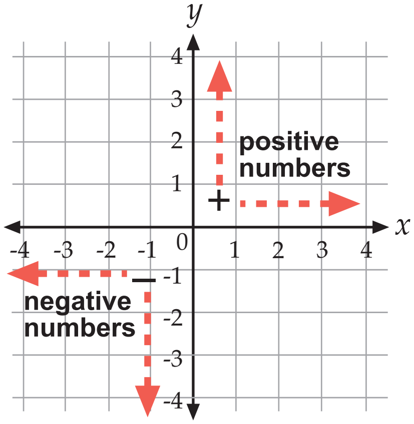
3. The *axes* divide the plane into four parts called **quadrants**. We start in the upper right and label that part Quadrant I. The upper left part is Quadrant II, the lower left Quadrant III, and the lower right Quadrant IV. The axes and the origin are not in any quadrant.

4. Each point in a coordinate plane can be represented by an **ordered pair** which describes the position of the point in relation to the *x*-axis and *y*-axis.

Coordinate Grid (or System)



Locating Points

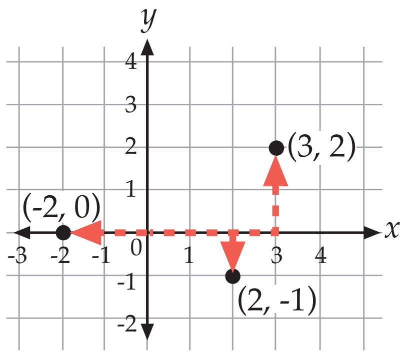


Let's look at the ordered pair (3, 2).

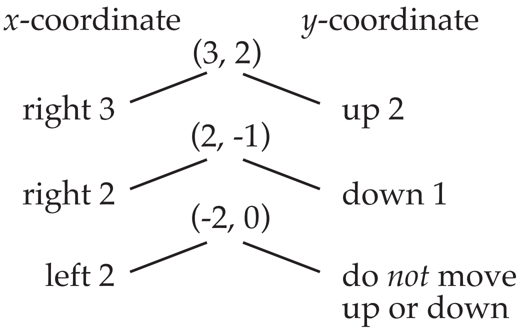
- Start at the *origin* (0, 0).
- The first number is the ***x*-coordinate**. The first number tells us whether to move left or right from the origin. If the number is *positive* we move *right*. If the number is *negative* we move *left*.
- The second number is the ***y*-coordinate**. The second number tells us whether to move up or down. If the number is *positive*, we move *up*. If the number is *negative*, we move *down*.

Study the following:

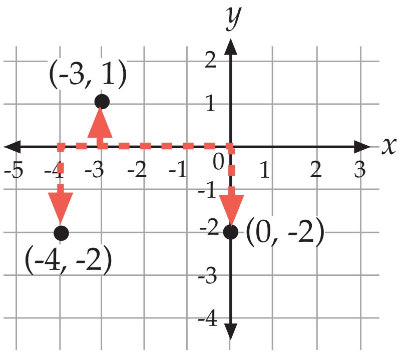
Graph (3, 2). Then graph (2, -1) and (-2, 0).



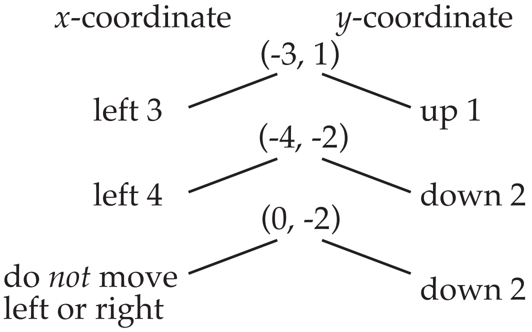
Start at the origin.



Graph (-3, 1). Then graph (-4, -2) and (0, -2).



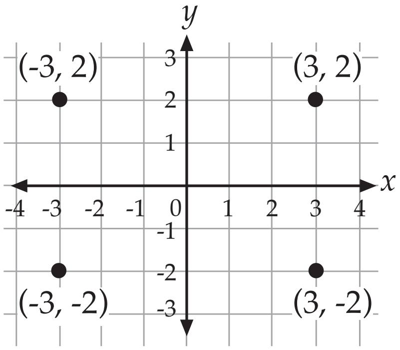
Start at the origin.



Let's Summarize:

When you graph a point, the signs of the coordinates tell which directions to move from the origin.

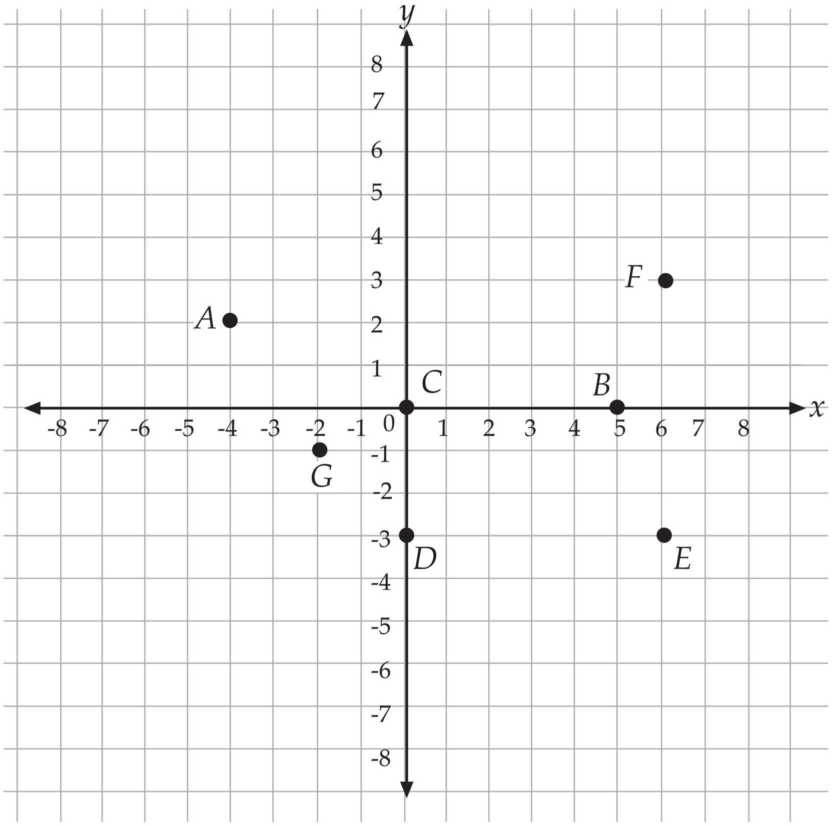
(3, 2)	(-3, 2)	(-3, -2)	(3, -2)
(+, +)	(- , +)	(- , -)	(+, -)
(right, up)	(left, up)	(left, down)	(right, down)



Note that the point (0, 0) is the origin.

Practice

Seven **points** *have been labeled on the* **coordinate grid** *below. Match each* **ordered pair** *with the* **correct point** *in the* **coordinate plane***. Write the letter of the point on the line provided.*



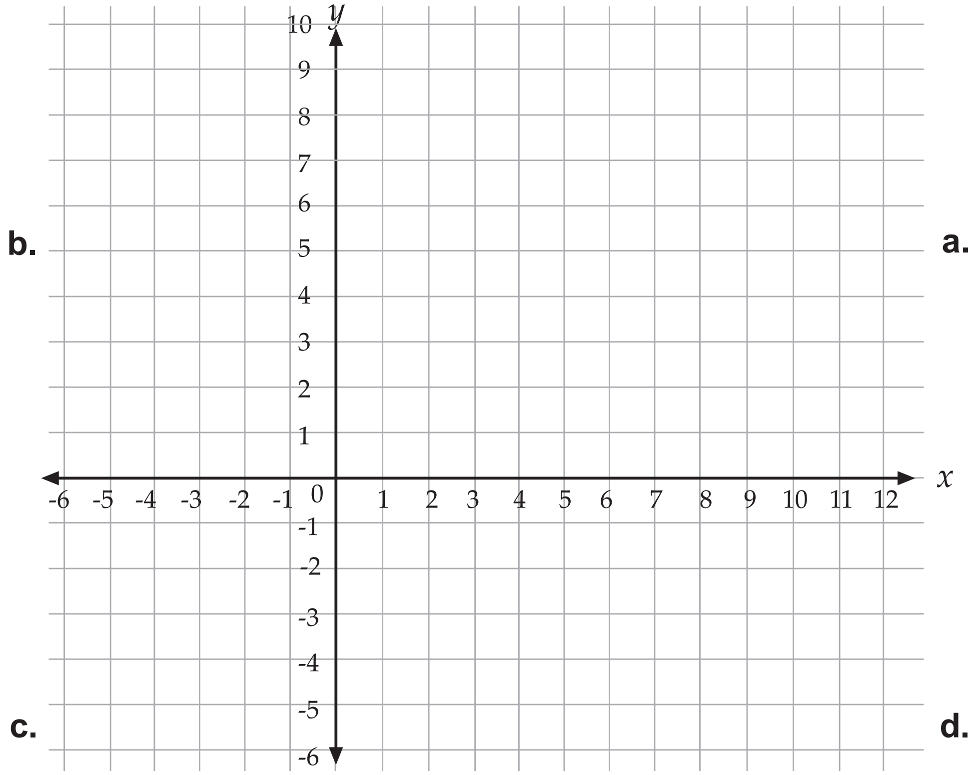
1. (5, 0) ____________
2. (6, 3) ____________
3. (0, 0) ____________
4. (-4, 2) ____________
5. (6, -3) ____________
6. (0, -3) ____________
7. (-2, -1) ____________

Answer the following.

8. What point(s) is (are) located in quadrant II? ____________
9. What point(s) is (are) located on the x-axis? ____________
10. What point(s) is (are) located on the y-axis? ____________

Complete the following.

11. On the coordinate system below, graph the three points and connect them with line segments. Classify the resulting triangle first by its sides (scalene, isosceles, equilateral), and then by its angles (obtuse, acute, right, equiangular).



a. (2, 1), (2, 5), (5, 1) ______________________

b. (-2, 1), (-4, 1), (-5, 4) ______________________

c. (-3, -1), (0, -3), (-6, -3) ______________________

d. (11, 0), (10, -5), (12, -5) ______________________

12. Using the same coordinate system that you used in number 11, plot these four points: (2, 7), (2, 9), (7, 9) and (7, 7). Connect the points so that a **rectangle** is formed. Label the rectangle number 12.

a. Find the width of the *rectangle*. ____________________________

b. Find the length of the rectangle. ___________________________

c. Find the **area** of the rectangle. _____________________________

Remember: The *area* of the rectangle can be found by multiplying the length (l) times the width (w).

$A = lw$

d. How many little **squares** are inside the rectangle? _________

Did you get the same answer that you did for number 12c

above? ____

Practice

Match each definition with the correct term. Write the letter on the line provided.

______ 1. the horizontal (↔) axis on a coordinate plane

______ 2. the vertical (↕) axis on a coordinate plane

______ 3. the graph of the intersection of the *x*-axis and *y*-axis in a coordinate plane, described by the ordered pair (0, 0)

______ 4. any of four regions formed by the axes in a rectangular coordinate system

______ 5. the point assigned to an ordered pair on a coordinate plane

______ 6. network of evenly spaced, parallel horizontal and vertical lines especially designed for locating points, displaying data, or drawing maps

A. coordinate grid or system

B. graph of a point

C. origin

D. quadrant

E. *x*-axis

F. *y*-axis

Practice

Use the list below to write the correct term for each definition on the line provided.

area (*A*)
axes (of a graph)
coordinates
intersection
ordered pair
***x*-coordinate**
***y*-coordinate**

________________________ 1. the point at which two lines meet

________________________ 2. the first number of an ordered pair

________________________ 3. the horizontal and vertical number lines used in a rectangular graph or coordinate grid system as a fixed reference for determining the position of a point

________________________ 4. the second number of an ordered pair

________________________ 5. numbers that correspond to points on a graph in the form (x, y)

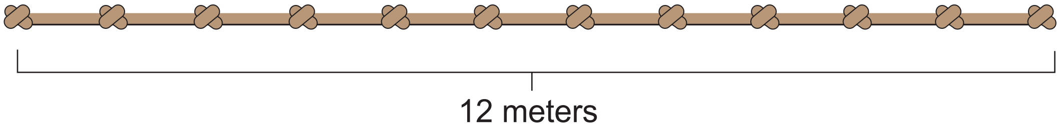
________________________ 6. the location of a single point on a rectangular coordinate system where the digits represent the position relative to the x-axis and y-axis

________________________ 7. the inside region of a two-dimensional figure measured in square units

Pythagorean Theorem

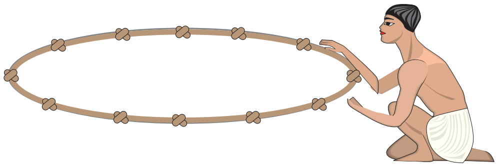
For thousands of years, concepts behind the **Pythagorean theorem** have been used. However, a Greek mathematician named Pythagoras is credited as the first person to write a proof of the theorem. Pythagoras lived during the sixth century B.C.

In about 2000 B.C., ancient Egyptian farmers reportedly used a 12-meter rope in surveying land. See the drawing below with knots equally spaced.

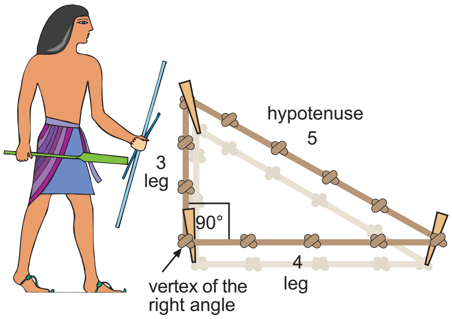


The farmers wanted to make square (90°) corners for their fields. They discovered a "magic 3–4–5" triangle that would help them do this.

- Workers took a rope, knotted it into 12 equal spaces, and formed it into a loop.



- Next they took three stakes and stretched the rope around them to make a triangle that had sides of 3, 4, and 5 units of equal space.

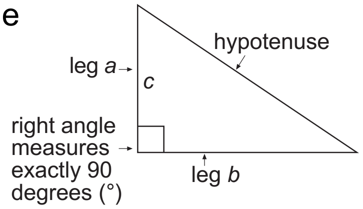


The stake supporting rope lengths of 3 and 4 turned out to be the *vertex* of a right angle, and a right triangle was formed. This method, in modified form, is still used by builders today. The sides forming the right angle are called **legs**, and the side opposite the right angle is the **hypotenuse**. The *hypotenuse* is always the longest side of a right triangle.

The ancient Greeks learned this trick from the Egyptians. Between 500 and 350 B.C., a group of Greek philosophers called the *Pythagoreans* studied the 3–4–5 triangle. They learned to think of the triangle's sides as the three sides of three squares. They generalized this to apply to *any* right triangle. This general statement became the *Pythagorean theorem*.

Pythagorean theorem: In a right triangle, the sum of the **squares** of the lengths of the legs, a and b, equals the square of the length of the hypotenuse c.

Algebraically: $a^2 + b^2 = c^2$



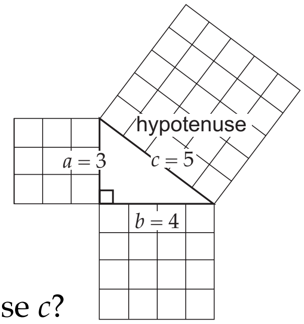
Remember: To *square* a number, multiply it by itself.
Example: The square of 6 or $6^2 = 6 \times 6 = 36$.

The Pythagorean theorem tells us that, in a right triangle, the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the two legs.

If the lengths of the legs are represented by a and b and the length of the hypotenuse is represented by c, then $a^2 + b^2 = c^2$.

Look at the figure at the right.

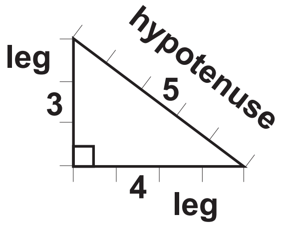
- What is the area (inside region) of the square on leg a?
- What is the area of the square on leg b?
- What is the area of the square on the hypotenuse c?



Is the sum of the areas on the legs equal to the area of the square on the hypotenuse?

We see that:

$$\begin{aligned} a^2 + b^2 &= c^2 \\ 3^2 + 4^2 &= 5^2 \\ 9 + 16 &= 25 \end{aligned}$$



Remember: *Area* (*A*) is the inside region of a two-dimensional figure and is measured in **square units**.

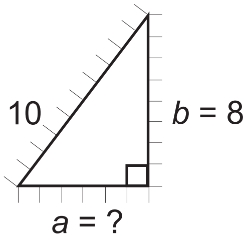
The Pythagorean theorem applies to all right triangles, not just the 3–4–5 right triangle. If you know the lengths of two sides of a right triangle, you can find the length of the hypotenuse or third side.

Example 1:

A right triangle has leg *b* with a unit measure of 8 and the hypotenuse with a unit measure of 10. What is the unit measure of leg *a*?

We can use the **formula** to determine the length of leg *a*.

$$\begin{aligned} a^2 + b^2 &= c^2 \\ a^2 + 8^2 &= 10^2 \\ a^2 + 64 &= 100 \\ a^2 + 64 - 64 &= 100 - 64\square \\ a^2 &= 36 \\ a &= \sqrt{36} \\ a &= 6 \end{aligned}$$



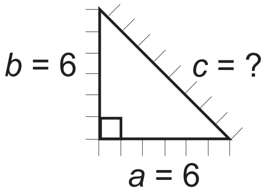
We see that *a* has a length of 6 units.

Example 2:

A right triangle has legs with a unit measure of 6. What is the measure of the hypotenuse?

We will use the same *formula* to determine the length of the hypotenuse.

$$\begin{aligned} a^2 + b^2 &= c^2 \\ 6^2 + 6^2 &= c^2 \\ 36 + 36 &= c^2 \\ 72 &= c^2 \\ \sqrt{72} &= c \\ 8.5 &\approx c \end{aligned}$$



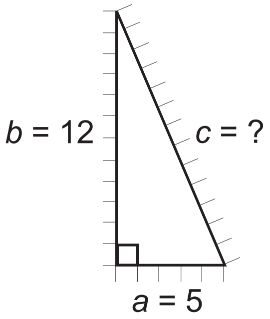
We see *c* has a length of about 8.5 units.

Note: Using our calculator, $\sqrt{72} = 8.485281374$ and **rounding** to the nearest tenth, is approximately equal to (≈) 8.5.

Example 3:

A right triangle has legs with unit measures of 12 and 5. What is the measure of the hypotenuse?

$$\begin{aligned} a^2 + b^2 &= c^2 \\ 5^2 + 12^2 &= c^2 \\ 25 + 144 &= c^2 \\ 169 &= c^2 \\ \sqrt{169} &= c \\ 13 &= \mathrm{c} \end{aligned}$$

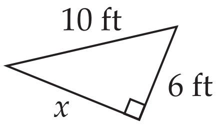


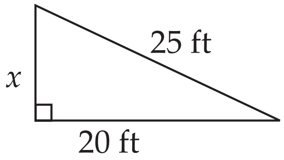
We see *c* has a length of 13 units.

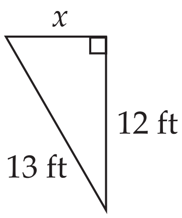
Practice

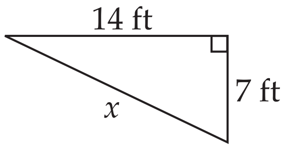
Use the **Pythagorean theorem** ($a^2 + b^2 = c^2$) *to solve each problem. Use a* **calculator** *or the* **square root table** *in* **Appendix A**. **Round answers to the nearest tenth.** *Show all your work.*

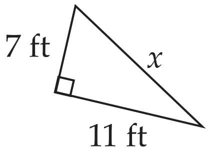
Find ***x*** *for the following.*

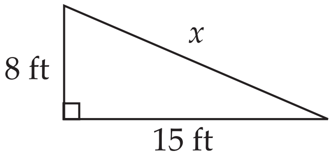
1. 

2. 

3. 

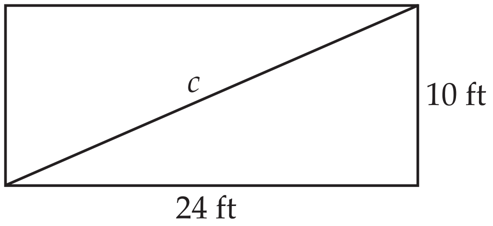
4. 

5. 

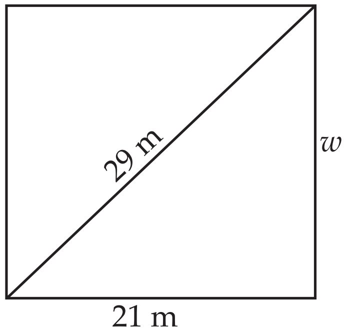
6. 

Solve the following. Show all your work.

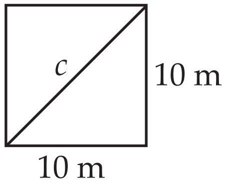
7. Find the length of the diagonal of the rectangle. ____________



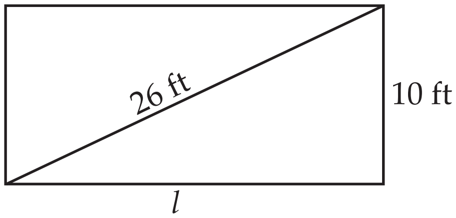
8. Find the width of the rectangle. ____________



9. Find the diagonal of the square. ____________



10. Find the length of the rectangle. ____________



Problem Solving with the Pythagorean Theorem

Earlier in this unit, we used similar triangles to indirectly measure the height of a flagpole. The Pythagorean theorem is frequently used for measuring objects indirectly.

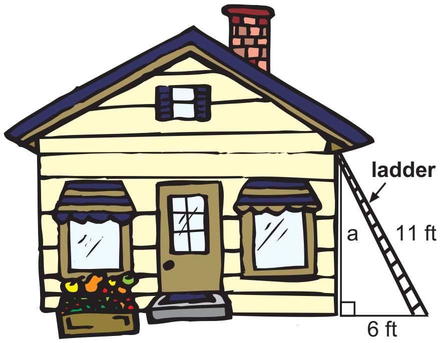
Example:

An 11-foot ladder is placed against a wall so that it reaches the top of the wall. The bottom of the ladder is 6 feet from the wall. How high is the wall? Round to the nearest whole number.

Write the Pythagorean theorem.

$$\begin{aligned} a^2 + b^2 &= c^2 \\ a^2 + 6^2 &= 11^2 \\ a^2 + 36 &= 121 \\ a^2 + 36 - 36 &= 121 - 36 \\ a^2 &= 85 \\ a &= \sqrt{85} \\ a &= 9.219544457 \\ a &\approx 9 \end{aligned}$$

Use a drawing.



Rounding to the nearest whole number, we find that the wall is approximately equal to (≈) 9 feet high.

Practice

Use the **Pythagorean theorem** ($a^2 + b^2 = c^2$) *to solve each problem.* **Illustrate** *the problem. Use a* **calculator** *or the* **square root table** *in* **Appendix A**. **Round answers to the nearest whole number.** *Show all your work.*

1. A boat travels 8 miles west and then 15 miles south. How far is it from its starting point?

   Answer: ____________

2. The bases on a softball diamond are 60 feet apart. How long is it from home plate to second base?

   Answer: ____________

3. A square, empty lot is 40 yards on a side. If Jason walks diagonally across the lot, how far does he walk?

   Answer: ____________

4. A cable is fastened to the top of an 80-foot television tower and to a stake that is 30 feet from the base of the tower. How long is the cable?

   Answer: ____________

5. A 13-foot ladder is placed against a wall. The bottom of the ladder is 5 feet from the base of the wall. How high up the wall does the ladder reach?

   Answer: ____________

6. A 12-foot rope is fastened to the top of a flagpole. The rope is anchored at a point on the ground 6 feet from the base of the flagpole. What is the height of the flagpole?

   Answer: ____________

7. An airplane is 4 miles directly above the airport runway. A person on the ground is 10 miles from the runway. How far is the person from the airplane?

   Answer: ____________

Practice

Use the list below to write the correct term for each definition on the line provided.

formula
hypotenuse
legs
Pythagorean theorem
square units
square (of a number)

________________________ 1. the square of the hypotenuse (*c*) of a right triangle is equal to the sum of the squares of the legs

________________________ 2. the longest side of a right triangle; the side opposite the right angle in a right triangle

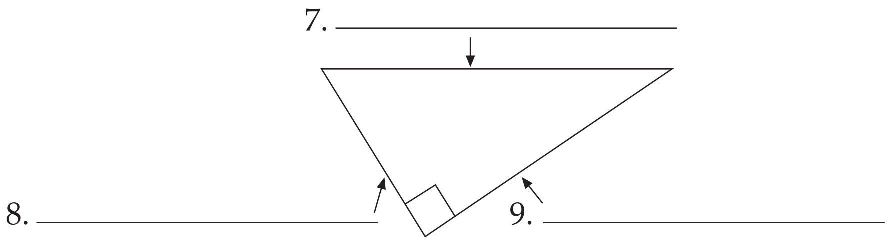
________________________ 3. in a right triangle, one of the two sides that form the right angle

________________________ 4. a way of expressing a relationship using variables or symbols that represent numbers

________________________ 5. the result when a number is multiplied by itself or used as a factor twice

________________________ 6. units for measuring area; the measure of the amount of an area that covers a surface

Label *the right triangle's* **legs** *and* **hypotenuse**.



Lesson Three Purpose

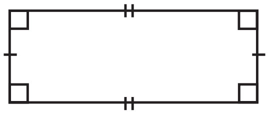
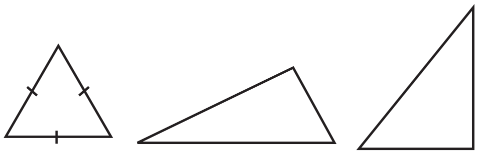
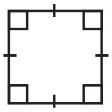
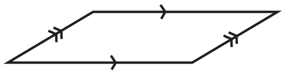
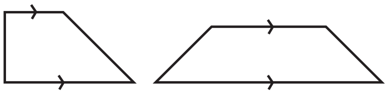
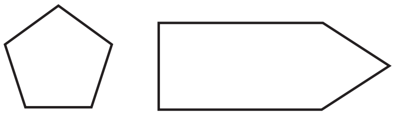
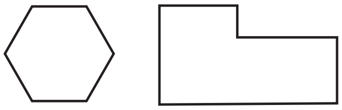
- Understand that numbers can be represented in a variety of equivalent forms, including integers, fractions, decimals, percents, scientific notation, exponents, radicals, and absolute value. (MA.A.1.4.4)
- Use estimation strategies in complex situations to predict results and to check the reasonableness of results. (MA.A.4.4.1)
- Use concrete and graphic models to derive formulas for finding perimeter, area, and circumference of two-dimensional shapes. (MA.B.1.4.1)
- Solve real-world and mathematical problems, involving estimates of measurements, including length, perimeter, and area and estimate the effects of measurement errors on calculations. (MA.B.3.4.1)
- Represent and apply geometric properties and relationships to solve real-world and mathematical problems including ratio and proportion. (MA.C.3.4.1)

Polygons

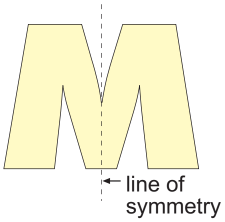
We have worked with rectangles, triangles, and squares in this unit. These shapes are considered *polygons* (closed shapes with straight sides). There are many other shapes which are included in the broad category of polygons.

This chart identifies many geometric shapes that are polygons.

Polygons

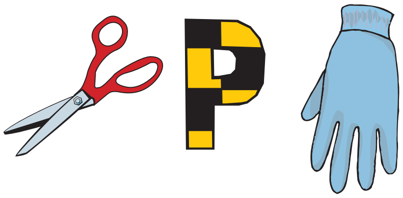
Name of Polygon	Description	Examples
rectangle	4 sides 4 right angles	
triangle	3 sides	
square	4 sides the same length 4 right angles (A square is also a rectangle.)	
parallelogram	4 sides 2 pairs of parallel sides (A rectangle is also a parallelogram.)	
trapezoid	4 sides 1 pair of parallel sides	
pentagon	5 sides	
hexagon	6 sides	

Symmetry



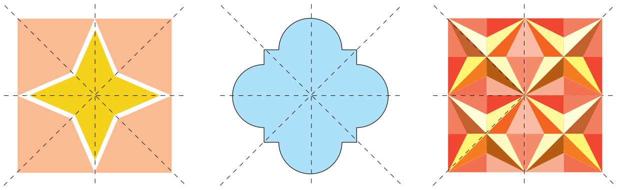
If a figure can be folded along a line so that it has two parts that are *congruent* and match exactly, that figure has *line symmetry*. Line symmetry is often just called *symmetry*. The *fold line* is called the **line of symmetry**. Sometimes more than one line of symmetry can be drawn.

A figure can have no lines of symmetry,



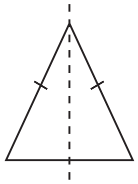
one line of symmetry,

or more than one line of symmetry.

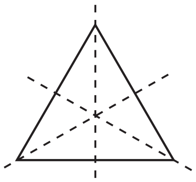


Consider the following polygons and their line(s) of symmetry.

isosceles triangle



equilateral triangle



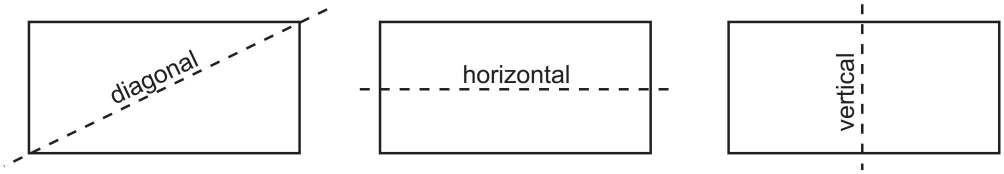
Notice three lines of symmetry.

Try this.

Which of these three dashed lines on the rectangles below create a line of symmetry?

- Get a sheet of paper and cut out three rectangles that are not squares.
- Make one fold on each rectangle as indicated by the dashed lines on the figures below.

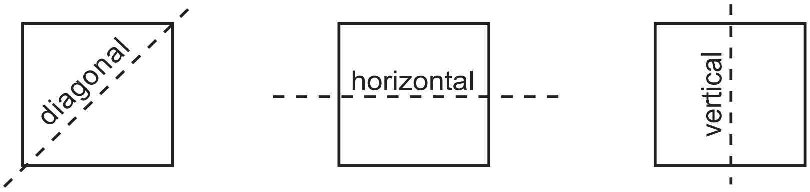
rectangle



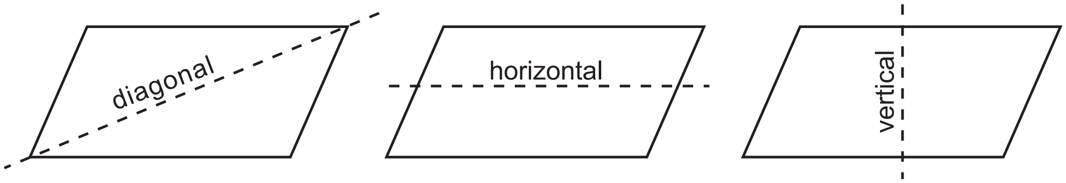
In each case, does one half of the folded paper fit exactly over the other half?

- Is the diagonal a line of symmetry? No.
- Do the horizontal and vertical lines create lines of symmetry? Yes.

If you had used a square, would your results be different? Yes, all three would result in lines of symmetry.



What if you used a parallelogram? Could you fold the parallelogram so that one half fits exactly over the other? No.

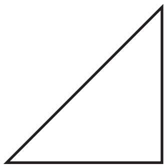


Practice

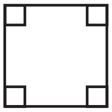
Use the chart on **polygons** *on page 462 and the list below to* **name** *each* **shape**. **One or more terms will be used more than once.**

hexagon
parallelogram
pentagon
rectangle
square
trapezoid
triangle

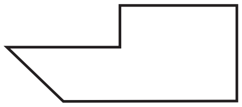
1. ____________________



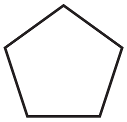
2. ____________________



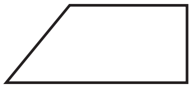
3. ____________________



4. ____________________



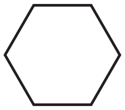
5. ____________________



6. ____________________

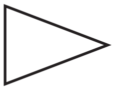
7. ____________________

8. ____________________



9. ____________________

10. ____________________



Answer the following.

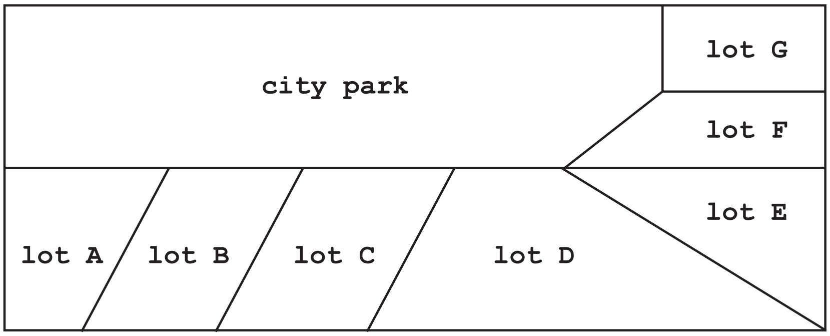
11. Which of the 10 shapes on the previous page can you draw one or more lines of symmetry? ______________________________

____

12. Select four polygons and draw their lines of symmetry.

Use the **map of a city block** *below to answer the following.*

Map of City Block



13. Which lot is a triangle? ______________________________

14. Which lot is a rectangle? ______________________________

15. Which lots are parallelograms? ______________________________

____

16. Which lots are squares? ______________________________

17. Which lots are trapezoids? ______________________________

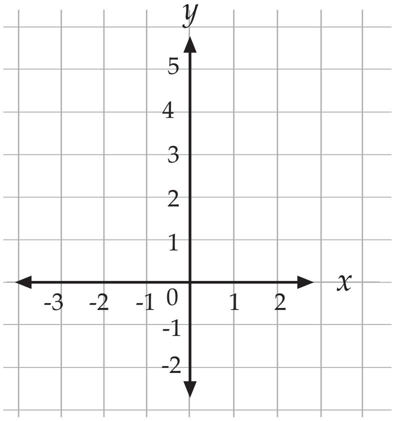
____

18. Which part of the city block is a pentagon? __________________

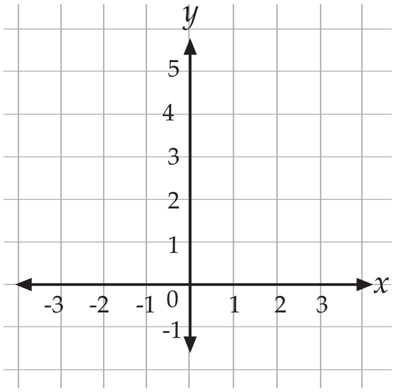
19. What is the shape of the city block? ______________________

Use the **coordinate grid** *below each problem to* **plot the ordered pairs** *listed.* **Connect the points** *as you plot them. When the shape is complete,* **draw** *any appropriate* **lines of symmetry**.

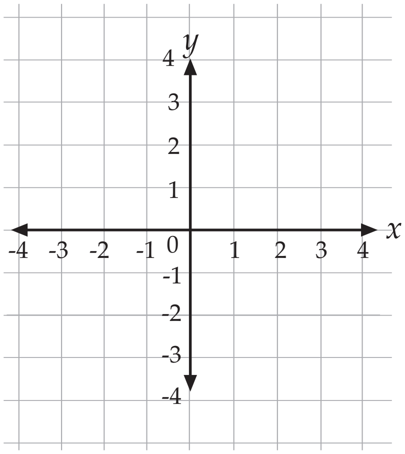
20. (-2, 1), (1, 1), (1, 4), (-2, 4), (-2, 1)



21. (3, 0), (0, 5), (-3, 0), (3, 0)



22. (-3, -1), (-3, 1), (-1, 1), (-1, 3), (1, 3), (1, 1), (3, 1), (3, -1), (1, -1), (1, -3), (-1, -3), (-1, -1), (-3, -1)



Use the **alphabet** *printed below to draw* **lines of symmetry** *through as many letters as possible.*

23. Which letters do *not* have symmetry?

____

A B C D E F G

H I J K L M N

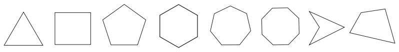
O P Q R S T U

V W X Y Z

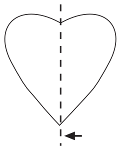
Practice

Match each definition with the correct term. Write the letter on the line provided.

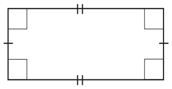
______ 1. a closed plane figure whose sides are straight and do not cross



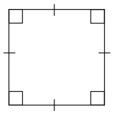
______ 2. a line that divides a figure into two congruent halves that are mirror images of each other



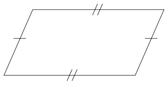
______ 3. a parallelogram with four right angles



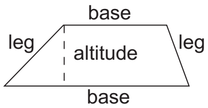
______ 4. a rectangle with four sides the same length



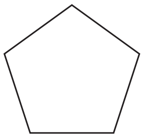
______ 5. a quadrilateral with two pairs of parallel sides



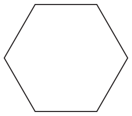
______ 6. a quadrilateral with just one pair of opposite sides parallel



______ 7. a polygon with five sides



______ 8. a polygon with six sides



A. hexagon

B. line of symmetry

C. parallelogram

D. pentagon

E. polygon

F. rectangle

G. square

H. trapezoid

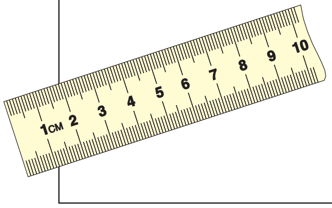
Perimeter of Polygons

To compute the **perimeter** (*P*) of a polygon we determine the distance around the polygon. Lengths of **sides** and various formulas are used. *Sides* are the edges of **two-dimensional** geometric figures. *Two-dimensional* figures have two dimensions: length (*l*) and width (*w*).

A brief review of conversions between metric units of length and U.S. customary measures of length may be helpful.

Common Conversion Factors

1 foot (ft) = 12 inches (in.)
1 yard (yd) = 3 feet = 36 inches
1 mile (mi) = 5,280 feet = 1,760 yards



1 centimeter (cm) = 10 millimeters (mm)
1 meter (m) = 100 centimeters
1 meter = 1000 millimeters
1 kilometer (km) = 1000 meters

Note: Appendix C has a list of conversions.

When we change from a *large* unit to a *smaller* unit, we *multiply* by the conversion factor.

	larger to smaller	conversion factor	multiply
a.	2 yd = ? ft	1 yd = 3 ft	2 x 3
b.	5 ft = ? in.	1 ft = 12 in.	5 x 12
c.	2 m = ? cm	1 m = 100 cm	2 x 100
d.	3 km = ? m	1 km = 1000 m	3 x 1000

Answers: (a) 6 ft (b) 60 in. (c) 200 cm (d) 3000 m

When we change from a *small* to a *larger* unit, we *divide* by the conversion factor.

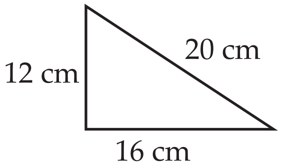
	smaller to larger	conversion factor	divide
e.	300 cm = ? m	100 cm = 1 m	300 by 100
f.	45 mm = ? cm	10 mm = 1 cm	45 by 10
g.	25000 m = ? km	1000 m = 1 km	25000 by 1000

Answers: (e) 3 m (f) 4.5 cm (g) 25 m

Formulas are used to find perimeter of triangles, squares, and rectangles. For the polygons in this unit, such as trapezoids and pentagons, we simply add the lengths of all the sides (*s*) to get perimeter (*P*).

Examples **Formulas**

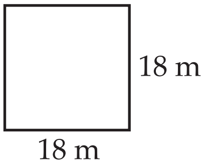
triangle



Perimeter = the sum of the lengths of the three sides—*a*, *b*, and *c*.

$P = a + b + c$
$P = 12 \text{ cm} + 16 \text{ cm} + 20 \text{ cm}$
$P = 48 \text{ cm}$
The perimeter is 48 centimeters.

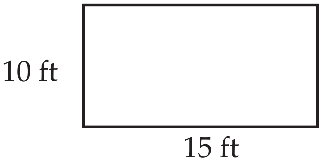
square



Perimeter = 4 times the length (*l*) of one side (*s* = side length).

$P = 4s$
$P = 4(18 \text{ m})$
$P = 72 \text{ m}$
The perimeter is 72 meters.

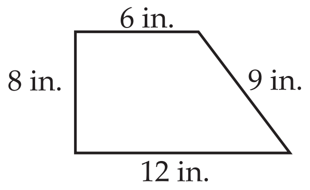
rectangle



Perimeter = 2 times the length (*l*) plus 2 times the width (*w*).

$P = 2l + 2w$
$P = 2(15 \text{ ft}) + 2(10 \text{ ft})$
$P = 30 \text{ ft} + 20 \text{ ft}$
$P = 50 \text{ ft}$
The perimeter is 50 feet.

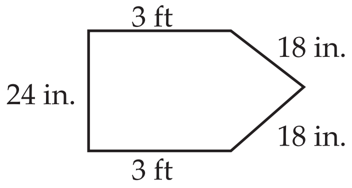
trapeziod



Perimeter = the sum of the lengths of all the sides.

$P = s_1 + s_2 + s_3 + s_4$
$P = 8 \text{ in.} + 6 \text{ in.} + 9 \text{ in.} + 12 \text{ in.}$
$P = 35 \text{ in.}$
The perimeter is 35 inches.

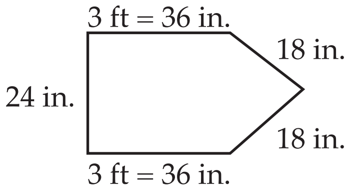
pentagon



Perimeter = the sum of the lengths of all the sides.
$P = s_1 + s_2 + s_3 + s_4 + s_5$

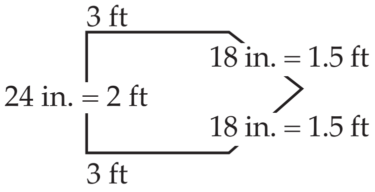
Alert: The units are *not* the same. Some are expressed in feet and others are expressed in inches. We *must* change feet to inches *or* inches to feet.

Example: Change *feet to inches*.



$P = 36 \text{ in.} + 18 \text{ in.} + 18 \text{ in.} + 36 \text{ in.} + 24 \text{ in.}$
$P = 132 \text{ in.}$
The perimeter is 132 inches.

Example: Change *inches to feet*.

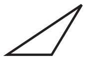
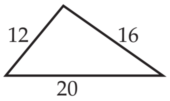
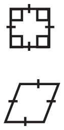
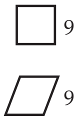
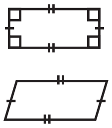
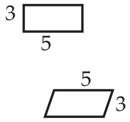
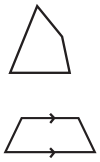
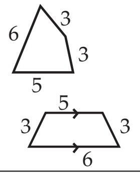
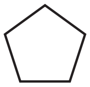
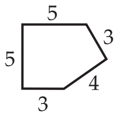
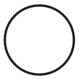
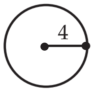


$P = 3 \text{ ft} + 1.5 \text{ ft} + 1.5 \text{ ft} + 3 \text{ ft} + 2 \text{ ft}$
$P = 11 \text{ ft}$
The perimeter is 11 feet.

So the perimeter of the pentagon above can be expressed as 132 inches *or* 11 feet. Either answer is correct.

Let's Summarize:

Perimeter (P) is the distance around a figure.

Mathematical Formulas for Perimeter (P)		
figure	**formula**	**example**
 **triangle**	$P = a + b + c$	 $P = 12 + 16 + 20$ $P = 48$ units
 **square** and **rhombus**	$P = 4s$	 $P = 4(9)$ $P = 36$ units
 **rectangle** and **parallelogram**	$P = 2l + 2w$	 $P = 2(5) + 2(3)$ $P = 16$ units
 **quadrilateral** and **trapezoid**	$P = s_1 + s_2 + s_3 + s_4$	 $P = 5 + 3 + 3 + 6$ $P = 17$ units
 **pentagon**	$P = s_1 + s_2 + s_3 + s_4 + s_5$	 $P = 5 + 5 + 3 + 4 + 3$ $P = 20$ units
 **circle**	$C = \pi d$ or $C = 2\pi r$	 $C \approx 2(3.14)(4)$ $C \approx 25.12$ units

Key

C = circumference

d = diameter

l = length

P = perimeter

r = radius

π = pi Use 3.14 or $\frac{22}{7}$ for π.

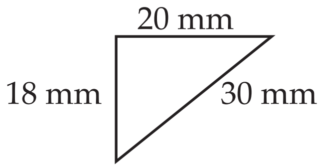
s = side

w = width

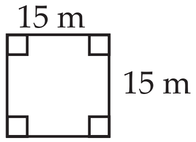
Practice

Find the **perimeter** *of each* **polygon**. *Refer to the conversion chart on page 470 if units are* not *the same.*

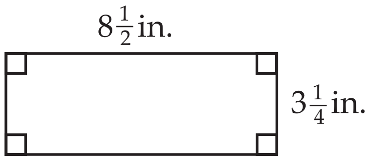
1. ____________________



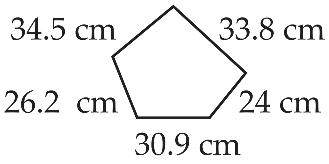
2. ____________________



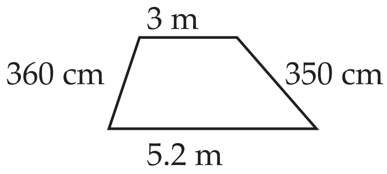
3. ____________________



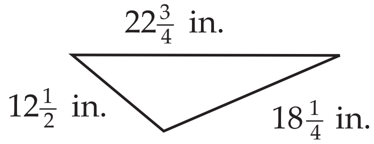
4. ____________________



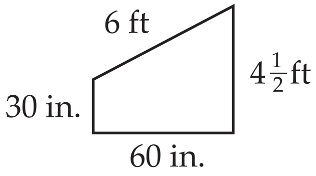
5. ____________________



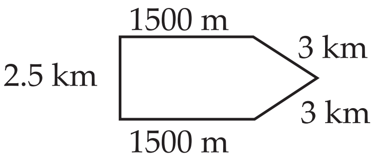
6. ____________________



7. ____________________



8. ____________________



Use the **appropriate formula** *to find the* **missing measure**. *Refer to* **formulas** *in unit as needed.*

9. square
s = 8 m

P = ____________

11. rectangle
l = 3.5 m
w = 1.8 m

P = ____________

10. rectangle
l = 12 cm
w = 6 cm

P = ____________

12. triangle
a = 13 m
b = 22 m
c = 12 m

P = ____________

---

13. rectangle
l = 9 km

w = ____________
P = 26 km

Hint: Use $P = 2(\text{length}) + 2(\text{width})$ and solve for w.
$26 = 2(9) + 2w$

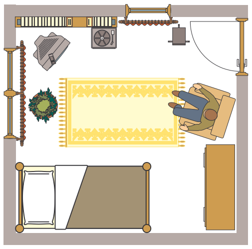
14. triangle
a = 250 in.
b = 168 in.

c = ____________
P = 562 in.

Hint: Use $P = a + b + c$ and solve for c.

15. The perimeter of a square room is 50 meters.

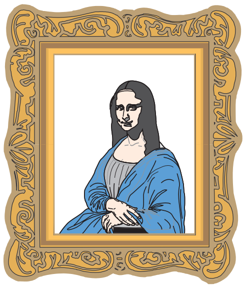
What is the length of each side of the room? ________________



P = 50 m

16. How many centimeters of framing are needed to frame a painting

that is 70 centimeters long and 55 centimeters wide? __________



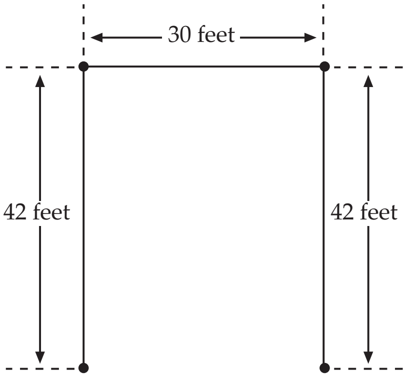
17. The perimeter of a triangular yard is 240 meters. The longest side is 110 meters and the shortest side is 50 meters.

How long is the third side? ______________________________

Answer the following.

18. Green Thumb Yard Design will plant viburnum shrubs in rows across the back and down two sides of the yard sketched below. At each of the 4 corners, a shrub will be planted. All shrubs will be planted 3 feet apart.

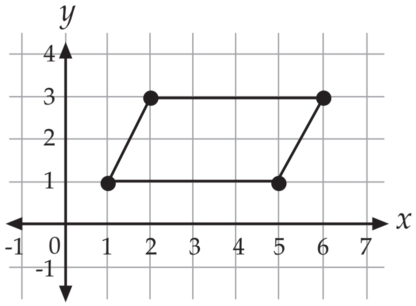
    How many shrubs need to be planted? ______________________



Finding Areas of Special Shapes

Finding the Area of Parallelograms

Remember, a parallelogram is a 4-sided figure with opposite sides parallel. Let's investigate how to find its area (A). *Area* is the inside region of a two-dimensional figure and is measured in *square units*. Let's take a piece of graph paper and plot these points (1, 1), (5, 1), (6, 3), and (2, 3). We will connect them with line segments so that a parallelogram is formed.

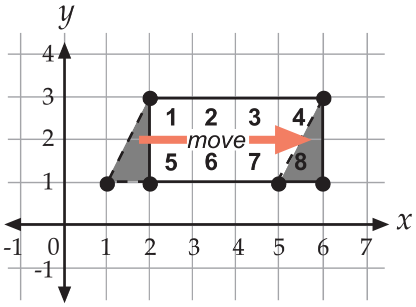


What do you think the area is of the parallelogram above?

Remember: Area is measured by the number of squares it takes to cover the region.

Let's look below and find the area.

In this case, there are 6 full squares, and then bits and pieces of other squares. What if we cut off the triangle formed by (1, 1), (2, 1), and (2, 3) and move it and place it on top of the triangle formed by (5, 1), (6, 1), and (6, 3). Notice the two triangles are congruent.



We now have a rectangle which has the same area as the parallelogram, and there are clearly 8 squares. The area of the parallelogram is 8 square units.

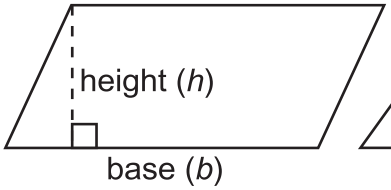
Remember: To find the area of a rectangle we must use the following formula and multiply the length times the width
$A = lw$.

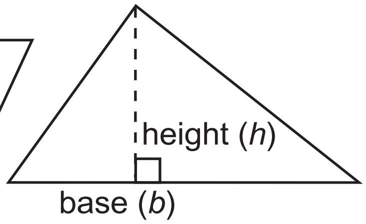
Any parallelogram can be *rearranged* to form a rectangle. Therefore, the formula for area of a parallelogram is closely related to the formula for area of a rectangle.

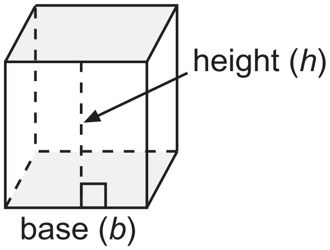
To find the area of a parallelogram we must multiply the **base (b)** times the **height (h)**.

$$A = bh$$

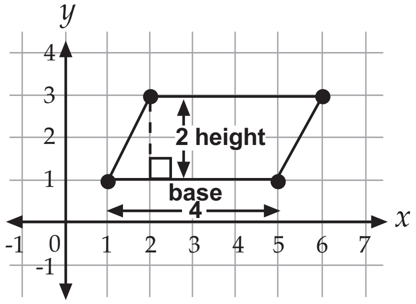
Let's make sure that we understand that the *base* (b) can be any side. The base is only the line or plane upon which a figure is thought of as resting. The *height (h)* is a little trickier. The height is the length of an **altitude**, a line segment perpendicular to the base (forming a right angle), drawn from the *opposite* side.





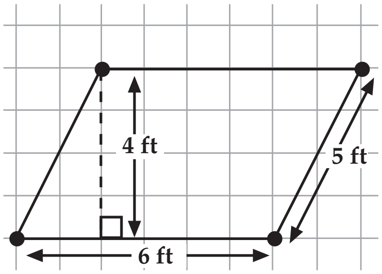


Look at the original parallelogram below. Can you see that the base is 4 units and the height is 2 units? Therefore the area is 8 square units.



Example:

Find the area of the parallelogram below.



Solution: $A = bh$

$A = (6)(4)$

$A = 24$ square feet

Note: The 5 was *not* used to find the area of the parallelogram because only the base and height are needed.

What is the perimeter of this parallelogram?

Remember: The perimeter of a polygon is the sum of all the sides.

Solution: $P = s_1 + s_2 + s_3 + s_4$

$P =$ 6 ft + 5 ft + 6 ft + 5 ft

$P =$ 22 ft

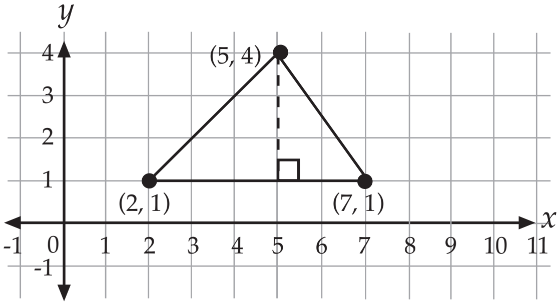
$P = 22$

Note: The 4 is *not* used to find the perimeter of the parallelogram because only the sides are needed.

Finding the Area of Triangles

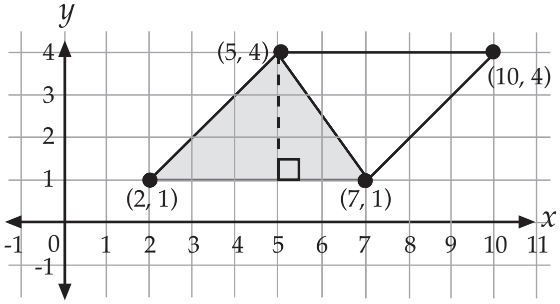
Challenge:

Plot these 3 points: (2, 1), (7, 1), and (5, 4) Connect them to make a triangle. Again it is difficult to be sure about the area.



Can you plot a 4th point to make it a parallelogram instead of a triangle?

Yes. The point is (10, 4).



We know that to find the area of the parallelogram we will use the following formula.

$$A = bh$$
$$A = (5)(3)$$
$$A = 15 \text{ square units}$$

What part of the parallelogram is the triangle? If you guessed $\frac{1}{2}$, you are correct.

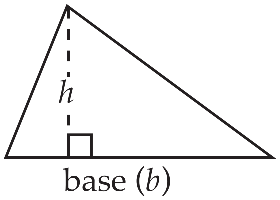
The area of our triangle = $\frac{1}{2}$(area of our parallelogram) = $\frac{1}{2}(15) = 7\frac{1}{2}$ square units.

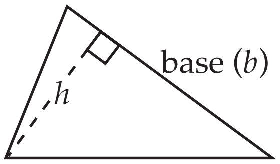
To find the area of a triangle, we use the following formula:

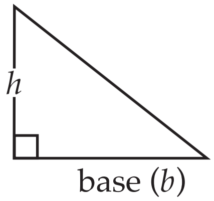
$$A = \frac{1}{2}bh$$

Remember: Any side of the triangle can be used as a base. The height is the length of the line segment draw from the *opposite* vertex perpendicular to the base (forming a right angle).

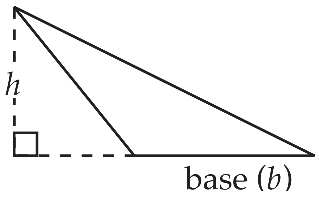
Study the following pictures.





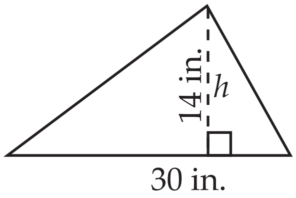


Note: Sometimes the height falls outside the triangle.



Example:

Find the area of the triangle below.



There are a few ways to do this problem.

$$A = \frac{1}{2}bh$$
$$A = \frac{1}{2}(30)(14)$$
$$A = (15)(14)$$
$$A = 210 \text{ square units}$$

$$A = \frac{1}{2}bh$$
$$A = \frac{1}{2}(30)(14)$$
$$A = \frac{1}{2}(420)$$
$$A = 210 \text{ square units}$$

$$A = \frac{1}{2}bh$$
$$A = (0.5)(30)(14)$$
$$A = 210 \text{ square units}$$

Hint:
$\frac{1}{2} = 0.5$

Remember: The **associative property** lets us group *without* changing the order. The **commutative property** lets us change the order.

Commutative Property and Associative Properties Revisited

You can add or multiply two numbers in any *order* without affecting the result. (See the **Rules for Order of Operations** chart in Unit 1.)

Addition	**Multiplication**
$a + b = b + a$	$a \bullet b = b \bullet a$
$3 + 7 = 7 + 3$	$3 \bullet 7 = 7 \bullet 3$

Think of the *commutative property* of addition and multiplication as a "sequence" or *order* making *no* difference in the outcome. For example, when getting dressed, you could put your shoes on *before* your belt.

However, the commutative property does *not* apply to subtraction and division.

$6 - 3 \neq 3 - 6$	$6 \div 3 \neq 3 \div 6$

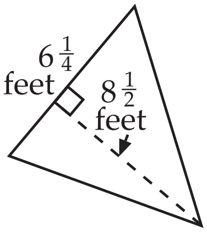
In other words, in this case, order *does* matter. You would *not* put your socks on *after* your shoes were already on your feet.

The *associative property* lets us add or multiply three or more numbers by grouping.

addition	multiplication
$(a + b) + c = a + (b + c)$	$(ab)c = a(bc)$

Example:

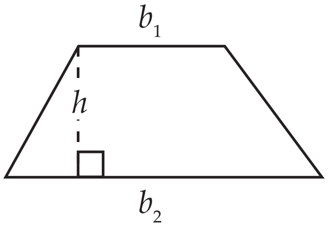
Find the area of the triangle below.



$$\begin{aligned} A &= \tfrac{1}{2}bh \\ &= \tfrac{1}{2}(6\tfrac{1}{4})(8\tfrac{1}{2}) \\ &= \tfrac{425}{16} \\ &= 26.6 \text{ square feet} \end{aligned}$$

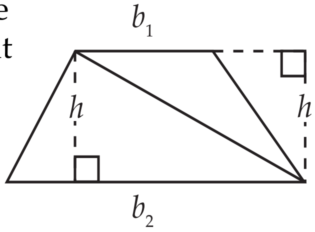
(rounded to nearest tenth)

How to Find the Area of a Trapezoid



Remember, a trapezoid is a four-sided figure with exactly two parallel sides, called *bases*. Since the bases aren't the same length, we denote them by using subscripts: b_1 (b sub one) and b_2 (b sub 2). **Note: These are not exponents!**

The height of a trapezoid is the length of a line segment drawn from one base perpendicular to the other base.



If we draw a diagonal from opposite corners, we see that we have two triangles with the same height, but different bases. The area of the trapezoid would be the sum of the areas of the two triangles.

$$\text{area of a trapezoid} = \tfrac{1}{2}b_1h + \tfrac{1}{2}b_2h$$

The **distributive property** would allow us to write the formula this way:

$$\text{area of a trapezoid} = \tfrac{1}{2} \bullet h(b_1 + b_2)$$

Distributive Property Revisited

You can *distribute* the numbers to write an *equivalent* or equal expression.

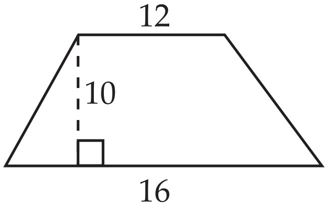
For all numbers a, b, and c:

$a(b + c) = ab + bc$
$3(7 + 4) = 3 \bullet 7 + 3 \bullet 4$

Think of the distributive property of multiplication as "spreading" things out or *distributing* things out to make the problem easier to work with—yet making *no* difference in the outcome.

Example:

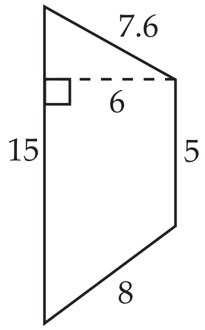
Find the area of the trapezoid below.



$$\begin{aligned} A &= \tfrac{1}{2} \bullet h(b_1 + b_2) \\ A &= \tfrac{1}{2}(10)(12 + 16) \\ A &= \tfrac{1}{2}(10)(28) \\ A &= 140 \text{ square units} \end{aligned}$$

Example:

Find the area of the trapezoid below.



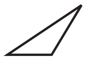
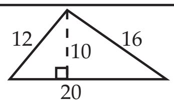
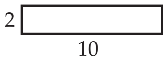
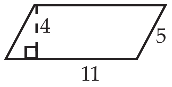
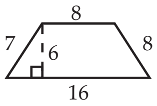
$$\begin{aligned} A &= \tfrac{1}{2} \bullet h(b_1 + b_2) \\ A &= \tfrac{1}{2}(6)(5 + 15) \\ A &= \tfrac{1}{2}(6)(20) \\ A &= \tfrac{1}{2}(120) \\ A &= 60 \text{ square units} \end{aligned}$$

Note: The measures of the other two sides do *not* matter!

Let's Summarize:

Area (A) is the number of square units a two-dimensional figure contains. Area is measured in square units.

Mathematical Formulas for Area (A)

figure	formula	example
 **triangle**	$A = \frac{1}{2}bh$	 $A = \frac{1}{2}(20)(10)$ $A = 100$ square units
 **rectangle**	$A = lw$	 $A = (2)(10)$ $A = 20$ square units
 **parallelogram**	$A = bh$	 $A = 11(4)$ $A = 44$ square units
 **trapezoid**	$A = \frac{1}{2}h(b_1 + b_2)$	 $A = \frac{1}{2}(6)(8 + 16)$ $A = 72$ square units

Key

A = area
b = base
h = height
l = length
w = width

Note: Appendix C contains a list of formulas.

Practice

Find the **area** *of the following figures using the grid below.*

1. ______________________

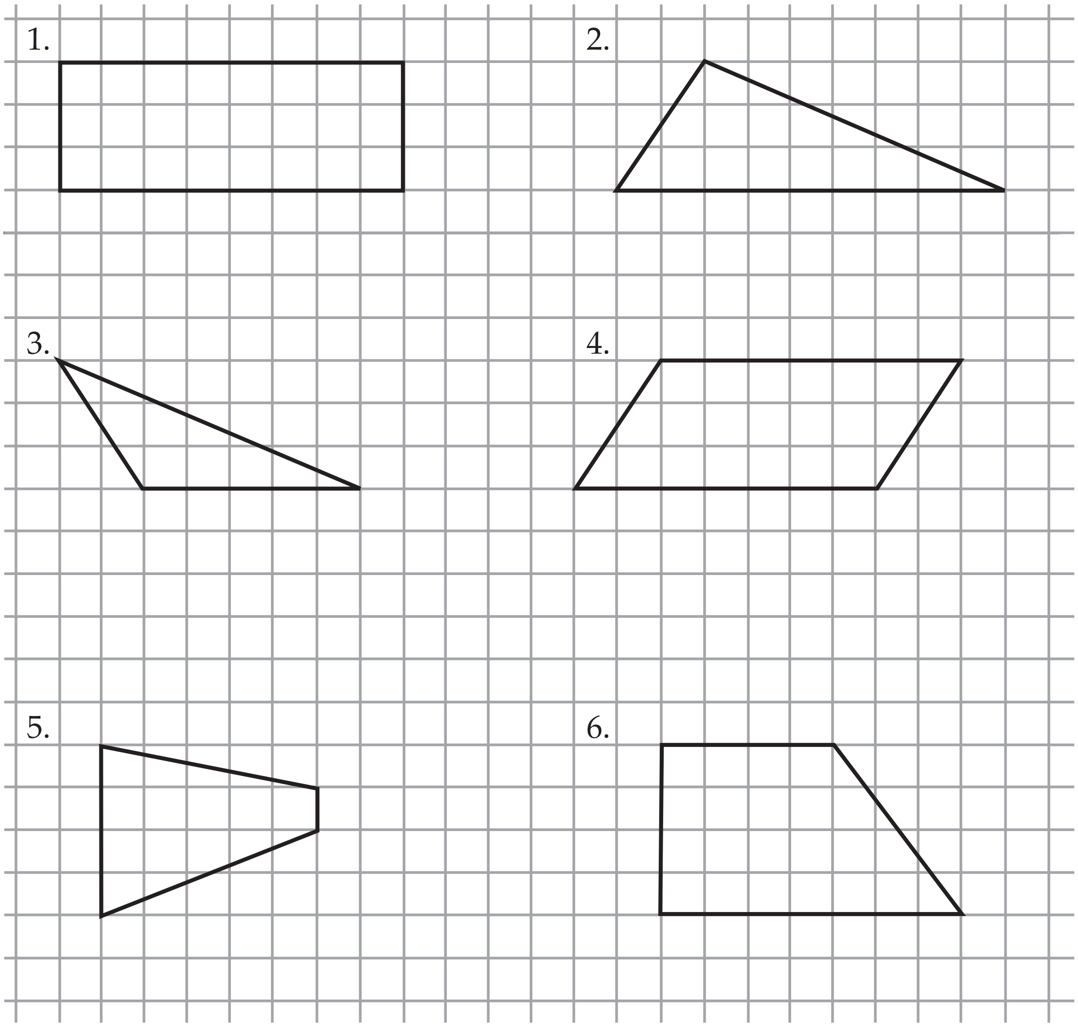
2. ______________________

3. ______________________

4. ______________________

5. ______________________

6. ______________________



Find the **area** *of the following. Refer to the* **formulas** *in this unit or in* **Appendix C** *as needed.*

7. A 4-inch square. ____________________

8. A rectangle with a length of $6\frac{1}{2}$ inches and a width of $3\frac{1}{4}$ inches.

   ____

9. A parallelogram with a base of $3\frac{1}{4}$ feet and a height of 4 feet.

   ____

10. A triangle with a base of 10 yards and a height of 7 yards.

    ____

11. A trapezoid with bases of $10\frac{1}{2}$ inches and $12\frac{1}{2}$ inches and a height of 4 inches. ____________________

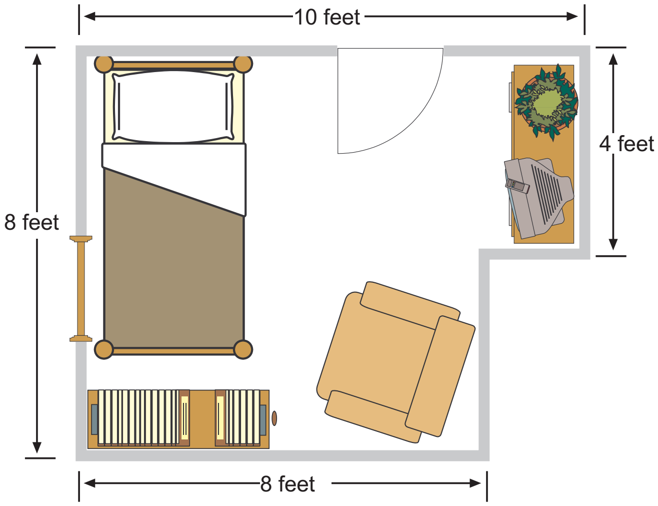
12. A trapezoid with bases of 0.3 kilometers and 0.5 kilometers and a height of 0.2 kilometers. ____________________

Practice

Answer the following. Refer to **formulas** *in unit or in* **Appendix C** *as needed.*

1. Here is a blueprint of Jamie's bedroom. She would like to carpet this room.
   (*Hint*: Can you divide this space into two rectangles?)

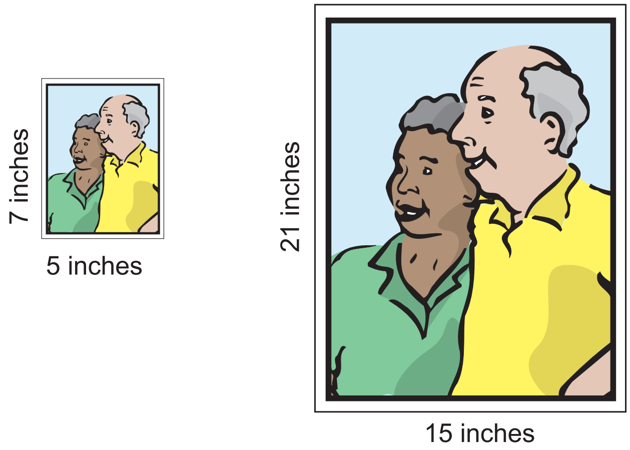
   Find the area. ____



She would also like to put a wallpaper border around the top of the room.

Find the perimeter. ___________________________________

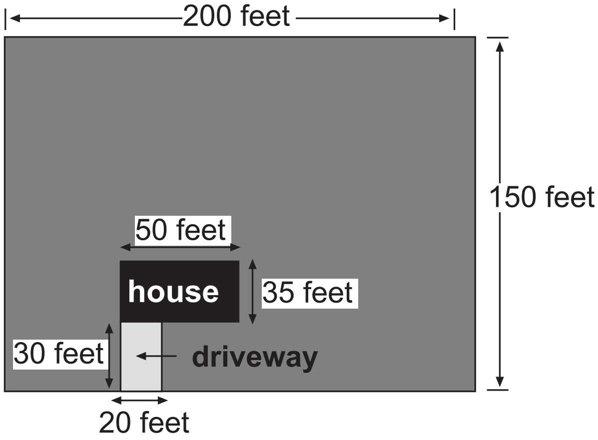
2. Latasha is having some pictures enlarged at a local copy store. She wants to enlarge a photo that is 5 inches by 7 inches so that the dimensions will be 15 inches by 21 inches.



How many times larger that the original photo will the area of the

new photo be? ______________________________

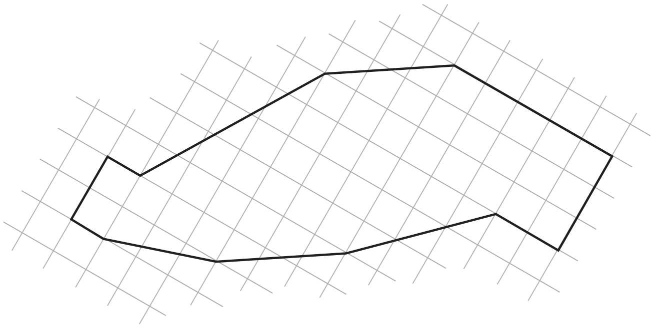
3. Tom wants to reseed his yard. A diagram of his lot is shown below. (**Hint:** Tom will not seed the house nor the driveway.)



How many square feet would he need to reseed his yard?

____

4. Below is an irregular shape. Find the area by dividing the shape into 5 smaller regions. ______________________

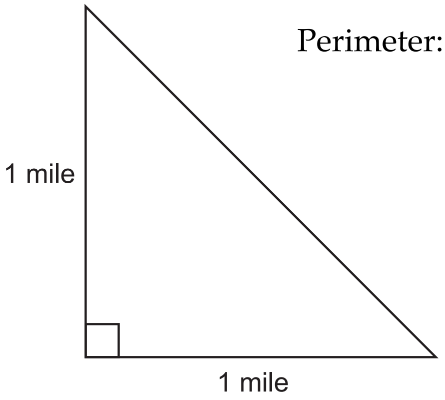


5. Find the area and perimeter of this right triangle. Use the Pythagorean theorem ($a^2 + b^2 = c^2$) and the square root table in Appendix A.

Round each answer to the nearest tenth.

Area: ______________________

Perimeter: ______________________



Practice

Use the list below to complete the following statements.

altitude
area (*A*)
associative property
base (*b*)
commutative property
distributive property
height (*h*)
perimeter (*P*)
sides (*s*)
two-dimensional
width (*w*)

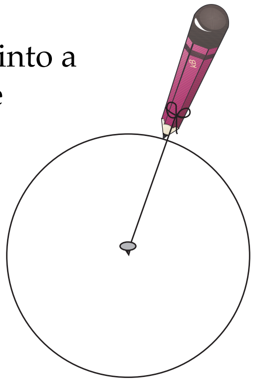
1. The ________________________ is the sum of the sides.
2. ________________________ are the edges of two-dimensional figures.
3. Figures that have length and width are ________________________ figures.
4. ________________________ is measured by the number of squares it takes to cover the region.
5. To find the area of a rectangle we must multiply the length times the ________________________ .
6. The ________________________ of a shape can be any side—it is the line or plane upon which a figure is thought to be resting.
7. The ________________________ is the length of a(n) ________________________ , a line segment perpendicular to the base, drawn from the *opposite* side.

8. The ________________________ lets us change the order without affecting the result.

9. Think of the ________________________ of multiplication as "spreading" things out to make it easier to work with—yet making *no* difference in the outcome.

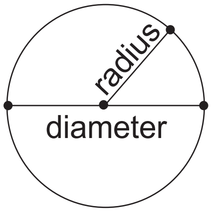
10. The ________________________ lets us group without changing the order and without affecting the result.

Circles



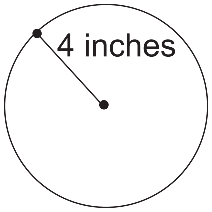
Suppose that you tie a string to a thumbtack and push it into a piece of cardboard. Next you tie a marker or pencil to the other end of the string and move the string around the thumbtack. The curved line drawn by the marker or pencil is a **circle**. The thumbtack is the **center of the circle**. The *center of the circle* is the point from which all the points on the circle are the same distance.

A *circle* is the set of all points that are the same distance from the center. The distance of any line segment drawn from the center to any point on the circle is called a **radius (*r*)**. A **diameter (*d*)** of a circle is the distance of a line segment from one point of the circle to another, through the center.



The *diameter* of a circle is *twice* its radius. $d = 2r$

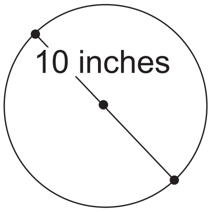
Example:



If the radius of this circle measures 4 inches, what is the diameter?

Solution: $d = 2r$
$d = 2(4)$
$d = 8$ inches

Example:

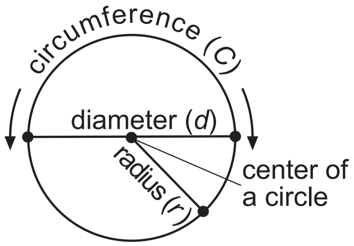


If the diameter of this circle measures 10 inches, what is the radius?

Solution:
$$\begin{aligned} d &= 2r \\ 10 &= 2r \\ 5 &= \mathrm{r} \end{aligned}$$

Circles have their own name for perimeter. The perimeter of a circle is called the **circumference (*C*)**. The *circumference* of a circle is the distance around the circle. The diameter is twice, or two times, the radius.

Circles



circumference (*C*) - the distance around a circle or the perimeter (*P*) of a circle

diameter (*d*) - a line segment that passes through the center of a circle to another point on a circle

center of a circle - the point from which all points on a circle are the same distance

radius (*r*) - any line segment from the center of a circle to a point on a circle

Practice

Complete the chart below.

A class measured the **diameters** and **circumferences** of several round objects. Here are their results.

- Find four more circular objects and the **circumference** and **diameter** of each.
- **Divide the circumference by the diameter**. Use a calculator and round each answer to the nearest hundredth.

Measurements of Round Objects

Item	Diameter	Circumference	Circumference Divided by Diameter
cookie tin	$5\frac{3}{4}$ inches	18 inches	3.13
auto tire	29 inches	91 inches	________
plate	9 inches	$28\frac{1}{4}$ inches	________
pail	10 inches	31.5 inches	________
________	________	________	________
________	________	________	________
________	________	________	________
________	________	________	________

Circumference

Centuries ago mathematicians found that the answer is always the same when a circumference is divided by its diameter. Your answers on the last practice were probably very close to 3.14.

$$\frac{C}{d} \approx 3.14$$

If we multiply both sides of this equation by d, we get the formula for finding the circumference of a circle.

$$C \approx 3.14d$$

The number 3.14 is represented by the Greek letter π, which is pronounced **pi**.

To find the circumference of a circle use the following formula:

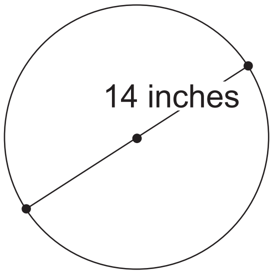
$C = \pi d$ Since $d = 2r$, we can also use $C = 2\pi r$.

$C = 2\pi r$ The commutative property allows us to rearrange the letters.

If you look on a scientific calculator, you will find a key for π. If you press it you will probably get 3.141592654.

π is neither 3.14 nor the long number which the calculator gave us. One of the great challenges faced by mathematicians through the ages was trying to find an *exact value* for π. However, there isn't one! The value of π is an *unending* decimal. In this section we will use 3.14 for π. Our answers will be *approximate* (≈)! Be aware that the fraction $\frac{22}{7}$ is also used for π.

Example:



Find the circumference of a circle whose *diameter* is 14 inches.

Solution: Since we know the diameter we will use this formula.

$C = \pi d$
$C = \pi(14)$ or
$C = 14\pi$ some people stop here
$C \approx 14(3.14)$
$C \approx 43.96$ inches The circumference is about 43.96 inches.

Example:

Find the circumference of a circle whose *radius* is 14 inches.

Solution: Since we know the radius we will use this formula.

$C = 2\pi r$
$C \approx 2(3.14)(14)$ multiply in any order
$C \approx 87.92$ inches The circumference is about 87.92 inches.

Example:

Jim bought a new bicycle. The wheel is 26 inches in diameter. How many inches will Jim go in just 1 revolution or turn of the bicycle tire?

Solution:
$C = \pi d$ 26 is the diameter of the bicycle wheel.
$C \approx (3.14)(26)$
$C \approx 81.64$ inches

To find the area of circle, use the following formula: $A = \pi r^2$

Example:

Find the area of a circle whose *radius* is 10 inches.

Solution:

$$\begin{aligned} A &= \pi r^2 \\ A &\approx (3.14)(10)^2 \\ A &\approx (3.14)(100) \\ A &\approx 314 \text{ square inches} \end{aligned}$$

Remember: Order of operations—square first

Example:

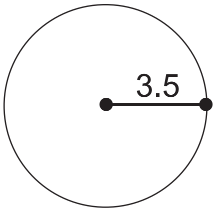
Find the area of a circle whose *diameter* is 10 inches.

Solution: The formula for area of a circle uses the radius, so we must first find the radius using what we know about the diameter.

$$\begin{aligned} d &= 2r \\ 10 &= 2r \\ 5 &= r \end{aligned}$$

$$\begin{aligned} A &= \pi r^2 \\ A &\approx (3.14)(5)^2 && \text{remember to square first} \\ A &\approx (3.14)(25) \\ A &\approx 78.5 \text{ square inches} \end{aligned}$$

Let's summarize how to do various problems using this circle.



Mathematical Formulas Diameter (*d*), Circumference (*C*), and Area (*A*)			
measurement	**formula**	**explanation**	**example**
diameter	$d = 2r$	*diameter* is twice the radius	$d = 2(3.5)$ $d = 7$
circumference	$C = \pi d$ *or* $C = 2\pi r$	*circumference* is distance around	$C \approx 2(\frac{22}{7} \cdot \frac{7}{2})$ *or* $2(3.14 \cdot 3.5)$ $C \approx 22$ square units
area	$A = \pi r^2$	*area* is squares on the inside	$A \approx (\frac{22}{7})(\frac{7}{2})^2$ *or* $(3.14)(3.5)^2$ $A \approx 38\frac{1}{2}$ square units

Key

A = area
C = circumference
d = diameter
r = radius
π = pi Use 3.14 or $\frac{22}{7}$ for π.
≈ = approximately equal to (e.g., 3.14 is an approximation for pi.)

Note: Appendix B has a list of mathematical symbols and their meanings and Appendix C has a list of formulas.

Practice

Fill in the chart below. **Round answers to the nearest hundredth.** *Refer to* **formulas** *in unit or in* **Appendix C** *as needed.*

Measurements

	Diameter	**Circumference**
1.	6 meters	________
2.	15 centimeters	________
3.	6.8 inches	________
	Radius	**Circumference**
4.	21 millimeters	________
5.	6 inches	________
6.	48 centimeters	________
	Radius	**Area**
7.	24 inches	________
8.	$3\frac{1}{2}$ yards	________
9.	$5\frac{1}{4}$ feet	________
	Diameter	**Area**
10.	28 inches	________
11.	72 yards	________
12.	126 inches	________

✓ **Check yourself:** Use the list of **scrambled answers** below and check your answers to problems 1, 4, 7, and 10 above

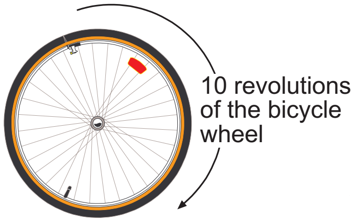
1,808.64 **131.88** **615.44** **18.84**

Practice

Answer the following. *Refer to the* **formulas** *in the unit or in* **Appendix C** *as needed.*

1. If the diameter of a circular flower bed is 28 feet, what is the circumference of the flower bed?

2. Each spoke in a wheel is 63 centimeters from the center to the outside of the rim. How long is each full rotation of the wheel? How far will this wheel move in 10 revolutions?



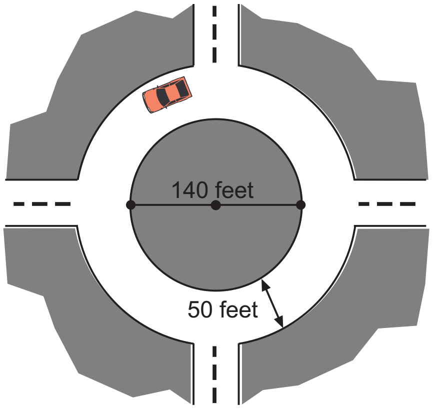
3. The diamete of a circular island on a roundabout is 140 feet, as shown below.

    - How far is it around the outside edge of this island?

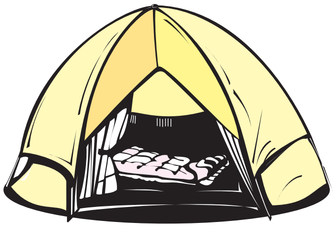
    Suppose that the street around the island is 50 feet wide.

    - What is the *outside* diameter measuring from one outer side of the roundabout to the other outer side?

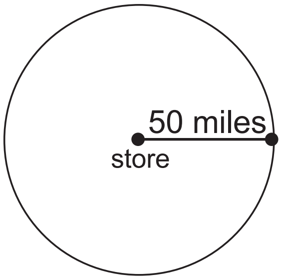
    - What is the circumference (disregarding the entrances) of the outside curb of the street?



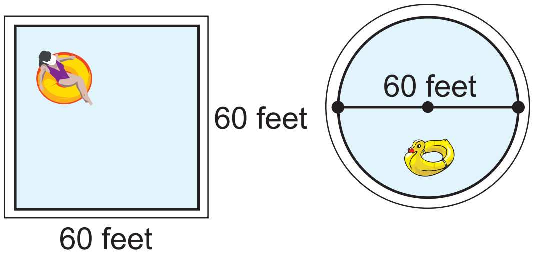
4. A circular tent has a diameter of 14 feet. How many square feet of ground space does it cover?



5. Mr. Brown, a merchant, says that some people drive 50 miles to shop at his store. How many square miles of territory lie inside the circle with center at his store and a radius of 50 miles?



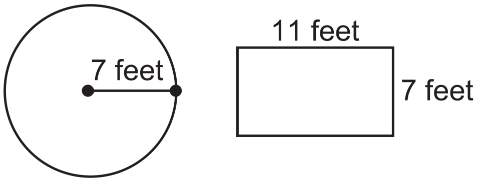
6. You have a choice of two different pools for your new pool. One is a square pool measuring 60 feet to a side. The other is a circular pool 60 feet in diameter. Find the area of each, and determine which pool is larger.



Circle the letter of the correct answer.

7. Compare the circle to the rectangle below. What is the ratio of the area of the circle to the area of the rectangle?

   a. 2:1
   b. 4:1
   c. 22:7
   d. 11:22



8. If the radius of a circle is 10 meters (m), then how long is the circumference?

   a. 10π m
   b. 100π m
   c. 20π m
   d. 250π m

Practice

Write **True** *if the statement is correct. Write* **False** *if the statement is* not *correct.*

__________ 1. A *circle* is the set of all points that are the same distance from the center.

__________ 2. The *center of the circle* is the point from which all points on the circle are the same distance.

__________ 3. The distance of any line segment drawn from the center to any point on the circle is called a *radius (r).*

__________ 4. A *diameter (d)* of a circle is the distance of a line segment from one point of the circle to another, through the center.

__________ 5. The diameter of a circle is four times its radius, $d = 4r$.

__________ 6. The *circumference (C)* of a circle is the distance around the circle.

__________ 7. The *perimeter* (P) of a circle is called the diameter.

__________ 8. When the circumference of a circle is divided by its diameter the answer is close to 3.14 or π.

__________ 9. The formula for finding the circumference of a circle is $C = \pi d$ or $C = 2\pi r$.

__________ 10. There is an *exact* value for pi and it is 3.141.

Lesson Four Purpose

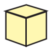
- Understand concrete and symbolic representations of real numbers in real-world situations. (MA.A.1.4.3)
- Use estimation strategies in complex situations to predict results and to check the reasonableness of results. (MA.A.4.4.1)
- Use concrete and graphic models to derive formulas for finding perimeter, area, surface area, circumference, and volume of two- and three-dimensional shapes, including rectangular solids, cylinders, cones, and pyramids. (MA.B.1.4.1)
- Solve real-world and mathematical problems, involving estimates of measurements, including length, perimeter, and area and estimate the effects of measurement errors on calculations. (MA.B.3.4.1)

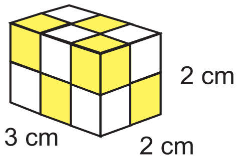
Volume

Volume of a Solid

Solid figures are **three-dimensional** figures that have length, width, and height and completely enclose a portion of space. *Solid figures* with flat surfaces are called **polyhedrons**. *Polyhedrons* are *three-dimensional* figures in which all surfaces are polygons. We will study two types of polyhedrons: **prisms** and **pyramids**.

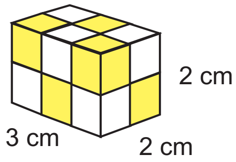
The **volume (*V*)** of a solid is the amount of space a solid contains. *Volume* is measured in **cubic units**. Therefore, the number of *cubic units* it takes to fill the solid is its volume.

Look at the solid figure with three **cubes**  across the front and two cubes on the end. We can think of the bottom layer as being made up of six cubes. There are two layers of cubes, so the *volume* (or number of cubes needed to fill the box) will be 12 cubic centimeters, abbreviated 12 cm^3.



A **rectangular prism** has six **faces**. A *face* is a flat surface of a solid figure. Each face on a *rectangular prism* is a rectangle.

This rectangular prism has a length of 3, a width of 2, and a height of 2.

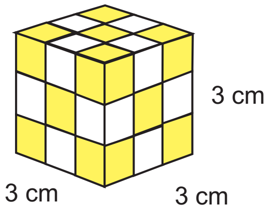


Example: *rectangular prism*

Volume = length x width x height

$V = lwh$
$V = (3)(2)(2)$
$V = 12 \text{ cm}^3$

If the solid has the same value for its length, width, and height, we have a cube.



Volume = $(3)(3)(3) = 27 \text{ cm}^3$

Volume of a Cylinder

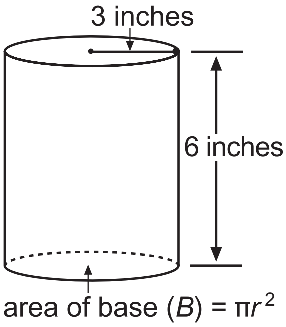
A **cylinder** is a three-dimensional figure with two parallel congruent circular bases.

To find the volume of a *cylinder*, we multiply the *area of the base* (B) by the height (h). Note that the base of a cylinder is a circle.

Example: *cylinder*

Volume = B x h (B is area of base)(h is height of cylinder)

$V = \pi r^2 h$
$V \approx (3.14)(3^2)(6)$
$V \approx (3.14)(9)(6)$
$V \approx 169.56$ cubic inches *or*
169.6 inches^3 *or* abbreviated as 169.6 in.^{3}
(rounded to the nearest tenth)



Volume of a Cone

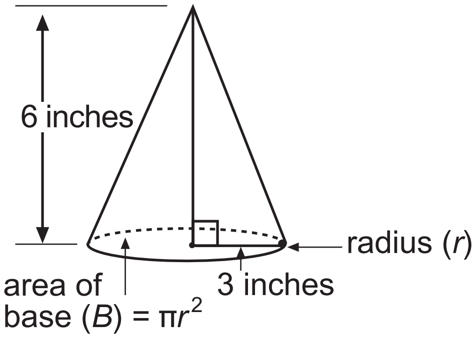
A **cone** is a three-dimensional figure with one circular base in which a curved surface connects the base to the vertex. A *right circular cone* has a center that forms a line perpendicular to its circular base. Consider a right circular cone with the same radius (r) and height (h) as the cylinder described on the previous page.

The cone's volume is $\frac{1}{3}$ of the volume of the above cylinder with a radius of 3 inches and a height of 6 inches. Therefore, to figure the cone's volume, we take $\frac{1}{3}$ of the volume of the cylinder.

Example: *cone*

Volume = $\frac{1}{3}$(volume of cylinder)

$$V = \frac{1}{3}\pi r^2 h$$
$$V = (\tfrac{1}{3})(\pi)(3^2)(6)$$
$$V = (\tfrac{1}{3})(\pi)(9)(6)$$
$$V = (\tfrac{1}{3})(54)(\pi)$$
$$V = 18\pi \text{ inches}^3$$



Frequently, answers are shown with the Greek letter (π) pi rather than calculating the decimal answer. This is useful when comparing areas and volumes of circular shapes. A decimal representation of our answer is

$$V \approx 18(3.14) \text{ inches}^3$$
$$V \approx 56.52 \text{ inches}^3 \textit{ or } 56.52 \text{ in.}^3$$

Volume of a Prism and Square Pyramid

A *prism* is a three-dimensional figure with congruent, polygonal bases and lateral faces that are all parallelograms. A *rectangular prism* is a six-sided prism whose faces are all rectangular.

A *pyramid* is a three-dimensional figure with a single polygonal base and triangular faces that meet at a common point (vertex). A **square pyramid** is a pyramid with a square base and four faces that are triangular.

A square pyramid's volume is $\frac{1}{3}$ of the volume of a prism with a square base and a height equal to a prism's height. See the drawings below. Note that the bases are squares with areas of 36 square inches. Heights of the prism and pyramid are 10 inches.

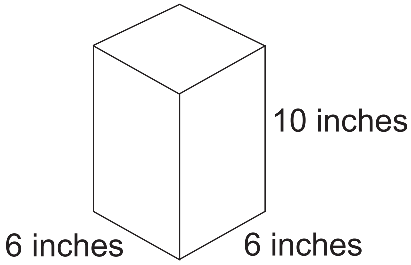
Example: *rectangular prism*

Volume = length x width x height

$V = (6)(6)(10)$

$V = (36)(10)$

$V = 360$ inches^3 (in.^{3})



Example: *square pyramid*

Volume $= \frac{1}{3}$(area of base) x h

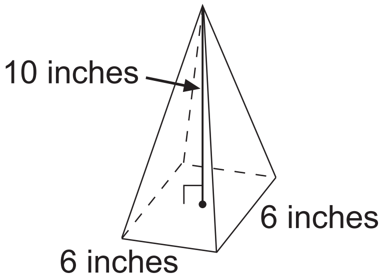
$V = \frac{1}{3}lwh$

$V = \frac{1}{3}(6 \text{ x } 6)(10)$

$V = \frac{1}{3}(36)(10)$

$V = 12(10)$

$V = 120$ inches^3 (in.^{3})



Volume of a Sphere

A **sphere** is a three-dimensional figure in which all points on the surface are the same distance from the center.

The volume of a *sphere* is calculated using the following formula.

Example: *sphere*

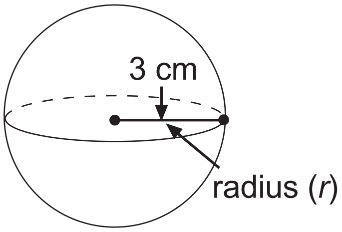
Volume $= \frac{4}{3}\pi r^3$

$V = \frac{4}{3}\pi(3^3)$

$V = \frac{4}{3}(27)\pi$

$V = 36\pi \text{ cm}^3$

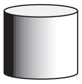
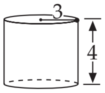
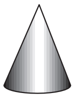
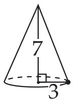
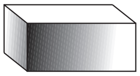
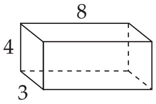
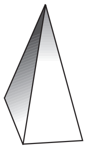
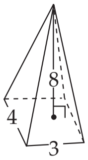
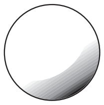
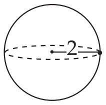
$(or \approx 113.04 \text{ cm}^3)$



Let's Summarize:

Volume (V) is the amount of space occupied in a three-dimensional figure. Volume is expressed in cubic units.

Mathematical Formulas for Volume (V)

figure	formula	example
 **right circular cylinder**	$V = \pi r^2 h$	 $V \approx (3.14)(3^2)(4)$ $V \approx 113.04$ cubic units
 **right circular cone**	$V = \frac{1}{3}\pi r^2 h$ *or* $V = \frac{1}{3}Bh$	 $V \approx \frac{1}{3}(3.14)(3)^2(7)$ $V \approx 66$ cubic units
 **rectangular solid**	$V = lwh$	 $V = 8(3)(4)$ $V = 96$ cubic units
 **square pyramid**	$V = \frac{1}{3} lwh$ *or* $V = \frac{1}{3}Bh$	 $V = \frac{1}{3}(3)(4)(8)$ $V = \frac{96}{3} = 32$ cubic units
 **sphere**	$V = \frac{4}{3}\pi r^3$	 $V \approx \frac{4}{3}(\frac{22}{7})(2^3)$ $V \approx 33\frac{11}{21}$ cubic units

Key

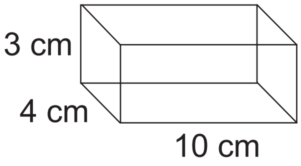
B = area of base
h = height
l = length
r = radius
V = volume
π = pi Use 3.14 or $\frac{22}{7}$ for π.
πr^2 = area of circular base

Note: Appendix C contains a list of various shapes with formulas for finding area, volume, and surface area. You will *not* need to memorize all of these formulas.

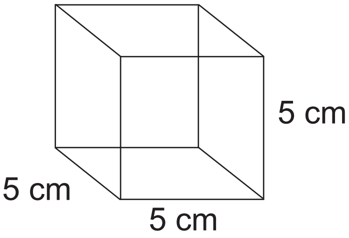
Practice

Find the **volume** *of each* **prism***. Use the formula* ***V* = *lwh***.

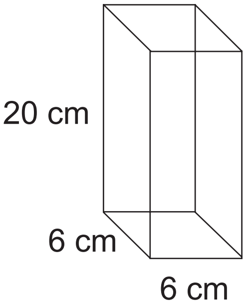
1.



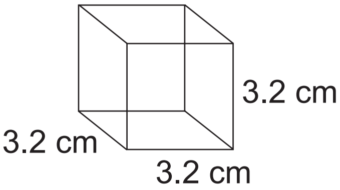
2.



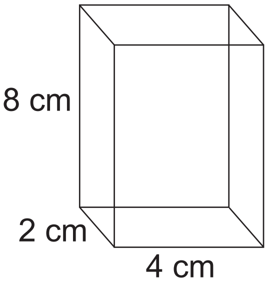
3.



4.



5.



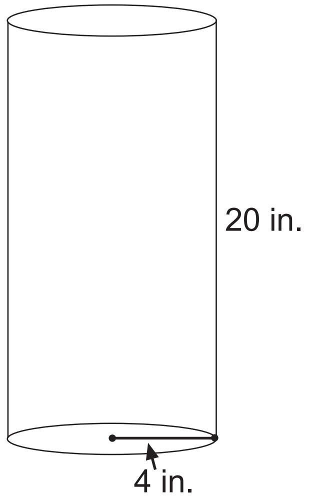
Find the **volume** *of each* **prism**. *Use the formula* $V = lwh$.

	length	**width**	**height**
6.	3 m	2 m	30 m
7.	10 ft	8 ft	15 ft
8.	16.2 cm	10 cm	1.5 cm
9.	$3\frac{1}{2}$ in.	2 in.	8 in.
10.	$12\frac{1}{2}$ m	8 m	3 m

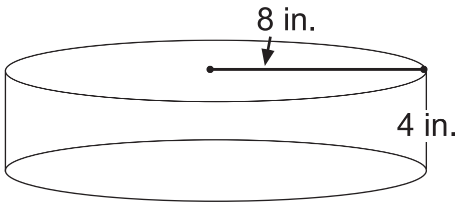
Practice

Find the **volume** *of each* **cylinder**. *Use the formula* $\mathbf{V = \pi r^2 h}$. **Round answers to the nearest tenth.**

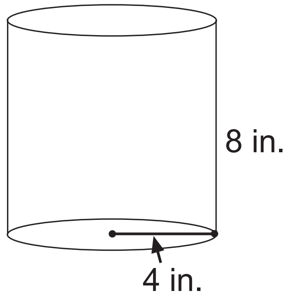
1.



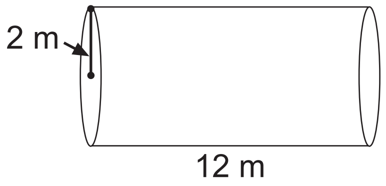
2.



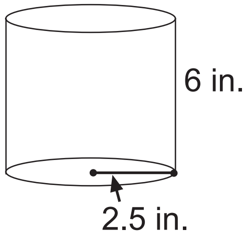
3.



4.



5.



Find the **volume** *of each* **cylinder**. *Use the formula* $V = \pi r^2 h$. **Round answers to the nearest tenth.**

6. radius, 2 in.; height, 10 in.; use 3.14 for π

7. diameter, 20 m; height 6 m; use 3.14 for π

   ✎ **Remember:** The radius is $\frac{1}{2}$ of the diameter.

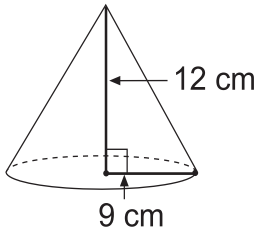
8. diameter, 5 ft; height 8 ft; use 3.14 for π

9. radius, 7 m; height 12 m; use $\frac{22}{7}$ for π

10. Refer to problems 2 and 3. Do the two cylinders have the same volume? ____________ If not, which volume is greater and by how much? ____

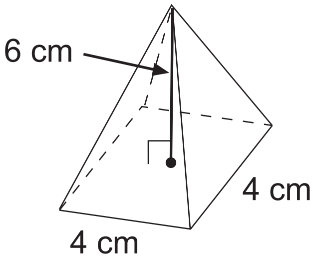
Find the **volume**. *Use the formula* $\frac{1}{3}\pi r^2 h$ *and* **round answer to the nearest tenth**.

11.



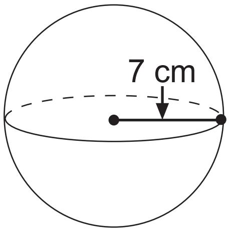
Find the **volume**. *Use the formula* $\frac{1}{3}Bh$ *and* **round answer to the nearest tenth**.

12.

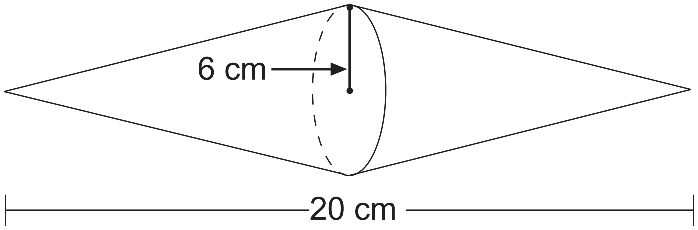


Find the **volume**. *Use the formula* $\frac{4}{3}\pi r^3$ *and* **round answer to the nearest tenth**.

13.



14. Two identical right circular cones have been placed with their bases touching to create the sculpture shown in the drawing below. The radius of each base is 6 cm and the total length of the object is 20 cm.

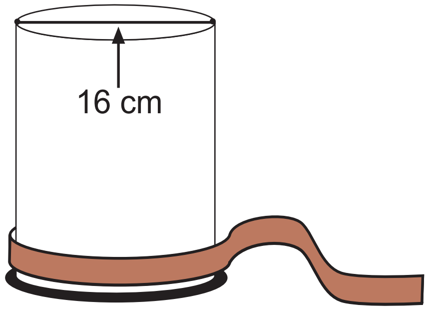


What is the volume, in cubic centimeters, of the sculpture?

Answer the following.

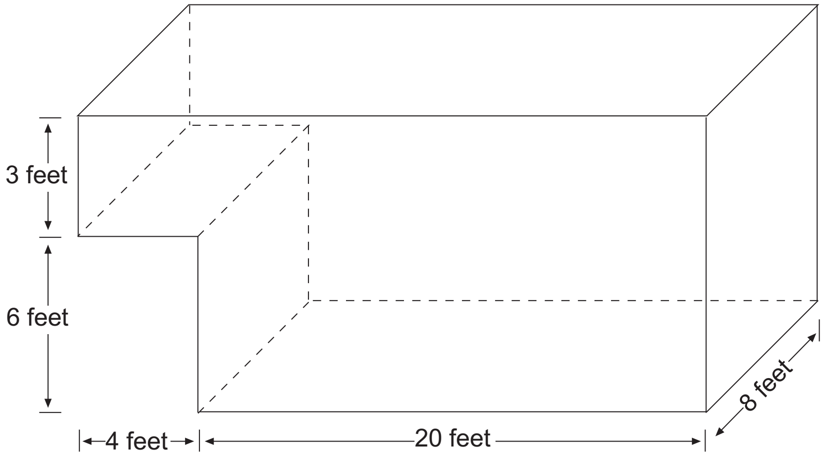
15. A cylindrical column 16 centimeters in diameter is strengthened by wrapping one steel band around the base of the column, with no overlap. What should be the length in centimeters of the steel band?

Remember: $C = d\pi$



Challenge Problem:

16. The diagram below shows the dimensions of the cargo area of a moving van. What is the maximum volume of cargo, in cubic feet, that can fit in the van?



Practice

Use the list below to complete the following statements.

circle
cone
cube
cubic units
cylinder
face
prism
rectangular prism
square pyramid
volume

1. The ______________________ of a solid is the amount of space a solid contains.

2. Volume is measured in ______________________ .

3. To find the volume of a ______________________ , we multiply the area of the base by the height.

4. Each base of a *cylinder* is a ______________________ .

5. A ______________________ is a rectangular prism that has six square faces.

6. A ______________________ has six faces.

7. A ______________________ is a flat surface of a solid figure.

8. A ______________________ is a three-dimensional figure with one circular base in which a curved surface connects the base to the vertex.

9. A ______________________ is a three-dimensional figure (polyhedron) with congruent, polygonal bases and lateral faces that are all parallelograms.

10. A ______________________ is a pyramid with a square base and four faces that are triangular.

Match each definition with the correct term. Write the letter on the line provided.

______ 11. three-dimensional figures that completely enclose a portion of space

______ 12. a three-dimensional figure in which all points on the surface are the same distance from the center

______ 13. a three-dimensional figure in which all surfaces are polygons

______ 14. existing in three dimensions; having length, width, and height

______ 15. a three-dimensional figure (polyhedron) with a single base that is a polygon and whose faces are triangles and meet at a common point (vertex)

A. polyhedron

B. pyramid

C. solid figures

D. sphere

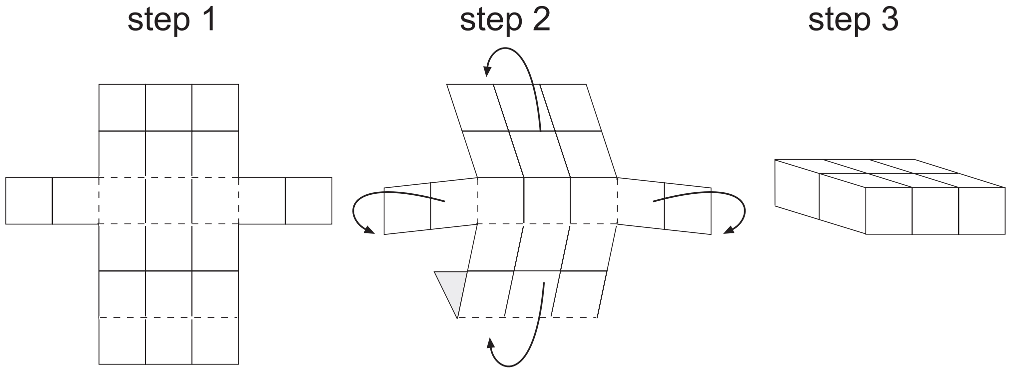
E. three-dimensional (3-dimensional)

Surface Area of Three-Dimensional Shapes

Using graph paper, we can see that two-dimensional shapes can be cut out and folded to create three-dimensional shapes. We call this two-dimensional pattern a **net**. A *net* is a plan which can be used to make a model of a solid.

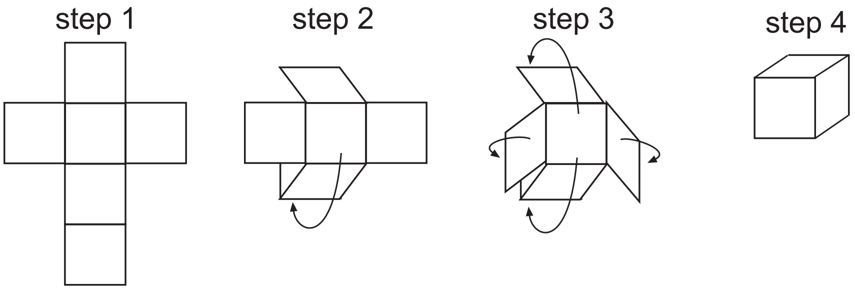
Example 1: *rectangular prism*

Draw this figure, cut it out, fold on the dashed lines, and tape it together to form a rectangular prism.



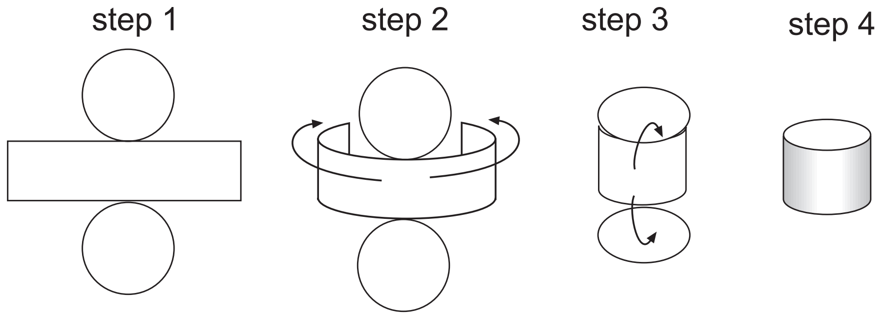
Example 2: *cube*

Draw this figure, cut it out, fold, and tape it together to make a cube.



Example 3: *cylinder*

Draw this figure, cut it out, wrap the rectangle around the circles, and tape it together to form a cylinder.



Surface Area and Lateral Area

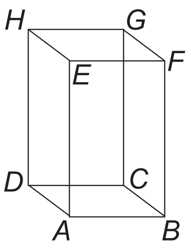
Surface area (*S.A.*) includes the areas of the faces of a figure that create a geometric solid. *Surface area* is expressed in *square units*.

The surface area of a rectangular prism is the sum of the areas of the bases and faces of the prism. Adding all the areas of the faces gives you the area of the entire surface of the box—the surface area.

Inspect the prism drawn below and we will identify its faces. There are the **lateral** faces (or *sides* of the shape), which are rectangles. *Lateral* refers only to the surfaces on the figure, not the *base*. Note the four rectangular faces. We can find the area of each face and add each of the four areas to get the *lateral area* (*L.A.*). Or we can use the perimeter of the bottom ($AB + BC + CD + DA$) times the height to calculate lateral area.

If the prism were wrapped in paper and we remove the paper, the shape would be a rectangle with length of $AB + BC + CD + DA$ and height AE (or BF or CG or DH). For total surface area, we find area of top and area of bottom and add these areas to our lateral area.

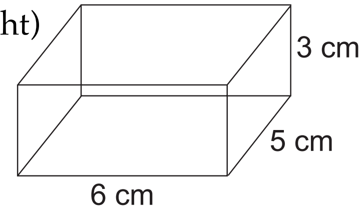
Our formula is
total surface area = 2(area of base) + perimeter of base(height).



Total Surface Area of a Rectangular Prism and a Cube

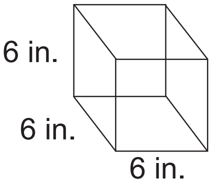
Example: *rectangular prism*

S.A. = 2(area of base) + perimeter of a base(height)
= 2(5 x 6) + (22)(3)
= 60 + 66
= 126 cm^2



Example: *cube*

S.A. = 2(area of base) + perimeter of base(height)
= 2(6 x 6) + 24(6)
= 72 + 144
= 216 in.^2

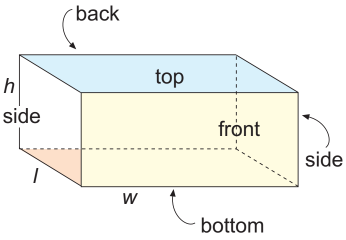


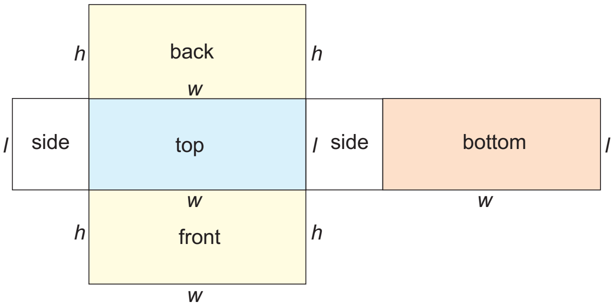
Another Way to Determine Surface Area of a Rectangular Prism and a Cube

Here's another way to determine the surface area of a prism. See the rectangular prism drawn below. Consider that a rectangular prism has six faces. You must find the area of each of its six faces:

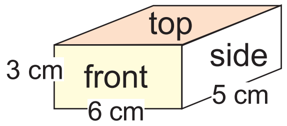
- a top and a bottom—2(lw)
- a front and a back—2(hw)
- two sides—2(lh)

$S.A. = 2lw + 2hw + 2lh$ or $2(lw + hw + lh)$

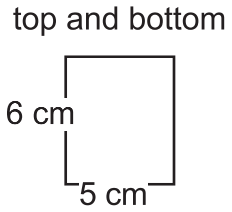


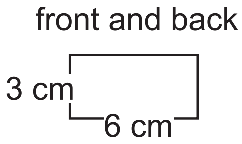


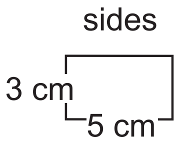
Example: *rectangular prism*



So, to find the surface area of the rectangular prism, you can make a sketch of the rectangular faces and label the dimensions.



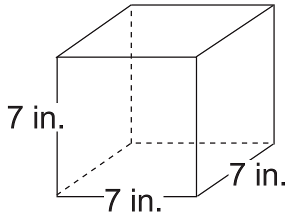




$S.A.$ = top and bottom + front and back + sides
$S.A. = 2lw + 2hw + 2lh$
$S.A. = 2(5 \bullet 6) + 2(6 \bullet 3) + 2(5 \bullet 3)$
$S.A. = 60 + 36 + 30$
$S.A. = 126\ \text{cm}^2$

Example: *cube*

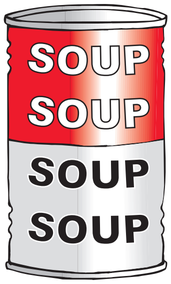
In a cube, all six faces have the same area. You only need to find the area of one face, then multiply by 6.

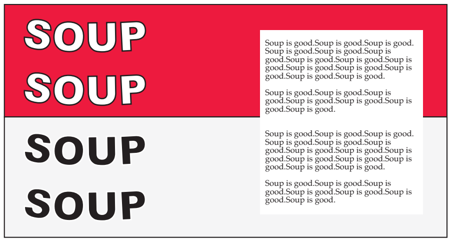


$S.A.$ = 6(side x side)
$= 6s^2$
$= 6(7 \bullet 7)$
$= 6(49)$
$= 294\ \text{in.}^2$

Total Surface Area of a Cylinder and a Sphere

To see an example of a cylinder, look at a can of soup. If we remove its wrapper by neatly cutting from top to bottom, the wrapper becomes a rectangle when we flatten it.



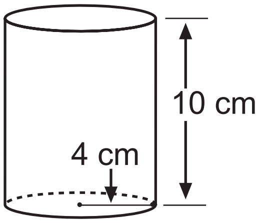


Instead of several rectangular shapes for the sides, our lateral area is a curved surface attached to a circular top and a circular bottom (the bases). Our total surface area formula is very similar to that of the prism except that our bases are circular.

Example: *cylinder*

The surface area of a cylinder is two times the circumference of the base (πr) times the height (h) plus two times the area of the base (πr^2).

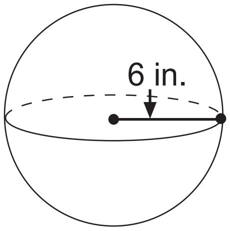
$$\begin{aligned} S.A. &= (2\pi r)(h) &&+ 2(\pi r^2) \\ &= 2\pi(4)(10) &&+ 2\pi(16) \\ &= 80\pi &&+ 32\pi \\ &\approx 251.2 &&+ 100.48 \\ &\approx 351.68 \text{ cm}^2 \end{aligned}$$



Example: *sphere*

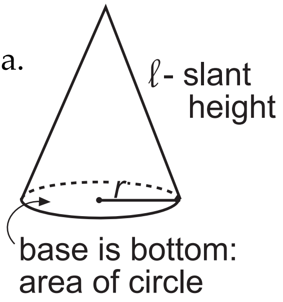
The surface area of a sphere is the circumference ($2\pi r$) times the diameter ($2r$) or ($2\pi r$)($2r$), which equals $4\pi r^2$.

$$\begin{aligned} S.A. &= 4\pi r^2 \\ &= 4\pi(36) \\ &= 144\pi \text{ in.}^2 \\ &\approx 452.16 \text{ in.}^2 \end{aligned}$$



Total Surface Area of a Cone and a Square Pyramid

To find surface area for cones and pyramids, notice that only one base and the outer edge are used in our formula. The outer edge is called the **slant height** and is designated by the letter ℓ.



Example: *cone*

The lateral area of a cone is half of the circumference of the base ($2\pi r$) multiplied by the slant height (ℓ).

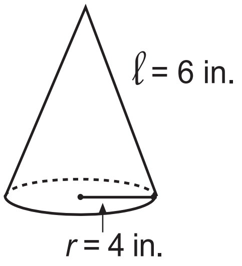
$$\text{lateral area} = \tfrac{1}{2}(2\pi r)(\ell)$$

To determine the surface area of a cone, you must find

- the lateral area $[\tfrac{1}{2}(2\pi r)(\ell)]$
- the area of the base (πr^2).

The total surface area of a cone is the lateral area of the cone $[\tfrac{1}{2}(2\pi r)(\ell)]$ plus the area of the base (πr^2).

$$\begin{aligned} S.A. &= \tfrac{1}{2}(2\pi r)(\ell) + \pi r^2 \\ &= \pi r \ell \quad + \pi r^2 \\ &= \pi(4)(6) \quad + \pi 4^2 \\ &= 24\pi \quad + 16\pi \\ &= 40\pi \text{ in.}^2 \\ &\approx 125.6 \text{ in.}^2 \end{aligned}$$



Example: *square pyramid*

The faces of a pyramid are triangles, so we use the *altitude* (or height) of a triangular face called the *slant height* (ℓ) in our formula. The lateral area of a pyramid is the sum of the areas of its lateral faces (not including the base). The slant height (ℓ) is the length of the altitude of a lateral face of a regular pyramid. To note the length of the edge of the square base, we use the symbol l.

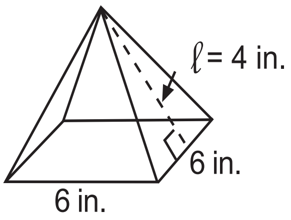
The lateral area of each triangular face is half of the length of the edge of the base (l) times the slant height (ℓ).

To determine the surface area of a square pyramid, you must find

- the lateral area of each triangular face ($\frac{1}{2}l\ell$)
- the area of the square base (l^2).

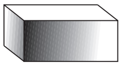
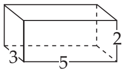
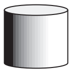
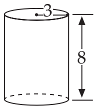
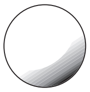
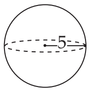
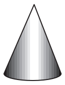
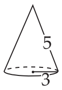
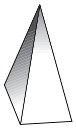
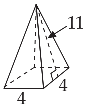
The surface area of a square pyramid is the area of four triangular faces [$4(\frac{1}{2}l\ell)$] plus the area of the base (ℓ^2).

$$\begin{aligned} S.A. &= 4(\tfrac{1}{2}l\ell) &&+ l^2 \\ &= 2l\ell &&+ l^2 \\ &= 2(6 \bullet 4) &&+ 6^2 \\ &= 2(24) &&+ 6^2 \\ &= 48 &&+ 36 \\ &= 84 \text{ in.}^2 \end{aligned}$$



Let's Summarize:

Surface area (*S.A.*) is the sum of all the areas of the faces (including the bases) of a solid figure. Surface area is expressed in square units.

Mathematical Formulas for Surface Area (*S.A.*)		
figure	**formula**	**example**
 **rectangular solid**	$S.A. = 2(lw) + 2(hw) + 2(lh)$	 $S.A. = 2(5 \bullet 3) + 2(3 \bullet 2) + 2(5 \bullet 2)$ $S.A. = 2(15) \quad + 2(6) \quad + 2(10)$ $S.A. = 30 \quad + 12 \quad + 20$ $S.A. = 62$ square units
 **right circular cylinder**	$S.A. = 2\pi rh + 2\pi r^2$	 $S.A. \approx 2(3.14)(3 \bullet 8) + 2(3.14)(3)^2$ $S.A. \approx 150.72 \quad + 56.52$ $S.A. \approx 207.24$ square units
 **sphere**	$S.A. = 4\pi r^2$	 $S.A. \approx 4(3.14)(5)^2$ $S.A. \approx (12.56)(25)$ $S.A. \approx 314$ square units
 **right circular cone**	$S.A. = \frac{1}{2}(2\pi r)\ell + \pi r^2$ $= \pi r\ell \quad + \pi r^2$	 $S.A. \approx (3.14)(3)(5) + 3.14(3)^2$ $S.A. \approx 47.1 \quad + 28.26$ $S.A. \approx 75.36$ square units
 **square pyramid**	$S.A. = 4(\frac{1}{2}l\ell) + l^2$ $= 2l\ell \quad + l^2$	 $S.A. = 2(4 \bullet 11) + 4^2$ $S.A. = 2(44) \quad + 4^2$ $S.A. = 88 \quad + 16$ $S.A. = 104$ square units

Key

h = height

l = length

r = radius

ℓ = slant height

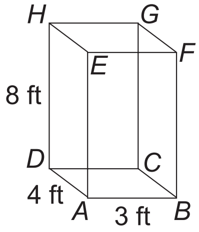
S.A. = surface area

π = pi Use 3.14 or $\frac{22}{7}$ for π.

Note: Appendix C has a list of formulas for surface area.

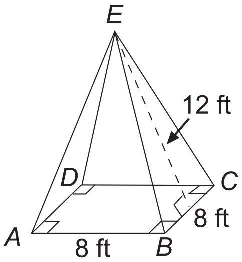
Practice

Use the examples **above** *each section to answer the items in that section. Refer to* **formulas** *in unit or in* **Appendix C** *as needed.*



Section 1

1. What kind of polygons are the lateral faces? ____________________
2. What kind of polygons are the bases? __________________________
3. What is the perimeter of the base? ______________________________
4. What is the height of the prism? ________________________________
5. If you were to paint the total surface of this prism, how many square feet (ft^2) need to be covered? __________________________________



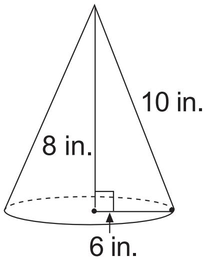
Section 2

6. What kind of polygon is the pyramid's base?__________________
7. Find the area of the base. ___________________________________
8. Name the lateral faces of the pyramid. _______________________

9. The area of the lateral surface is 4 times $\frac{1}{2}$(length of the base) *or*

2(length of the base) x ______________________________ .

10. Find the total surface area. ______________________________



Section 3

11. What is the slant height of the cone? ____________________

12. Find the area of the base. Leave π in your answer.

____

13. Find the perimeter of the base. Again, leave π in your answer.

____

14. Total surface area = lateral area + area of the base

$$= \frac{1}{2}(2\pi r)(\ell) \quad + \pi r^2$$
$$= \pi r\ell \quad + \pi r^2$$
$$= \pi(6)(10) \quad + \pi(6)^2$$
$$= 60\pi \quad + 36\pi$$
$$= 96\pi \text{ square inches or } 96\pi \text{ in.}^2$$

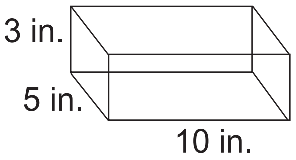
$\approx$ ____________ in.^{2}

Round to the nearest tenth.

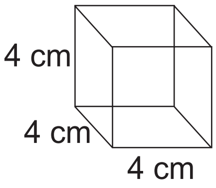
Practice

Find the total **surface area** *of the following* **solids**. *For problems using π,* **round answers to the nearest tenth**. *Refer to formulas in unit or in* **Appendix C** *as needed.*

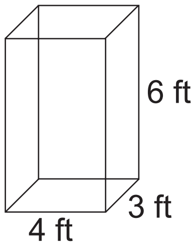
1.



2.



3.



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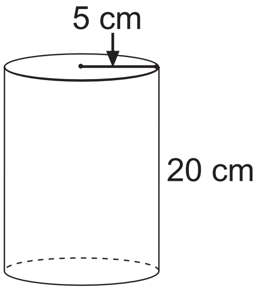
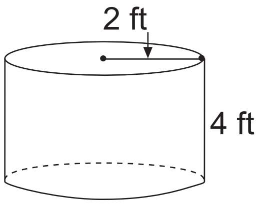
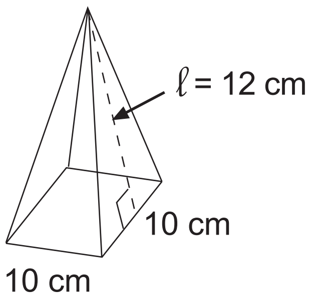


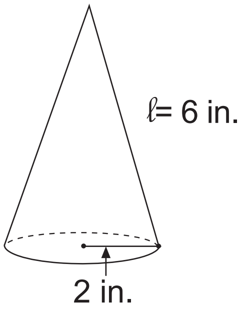
5.



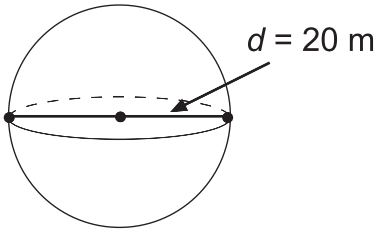
6.



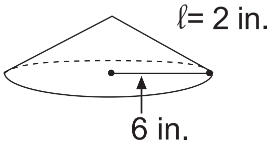
7.



9.



8.



Match each definition with the correct term. Write the letter on the line provided.

______ 10. the shortest distance from the vertex of a right circular cone to the edge of its base; the perpendicular distance from the vertex of a pyramid to one edge of its base

______ 11. a surface on the side of a geometric figure, as opposed to the base

______ 12. the sum of the areas of the faces of the figure that create the geometric solid

______ 13. a plan which can be used to make a model of a solid; a two-dimensional shape that can be folded into a three-dimensional figure

A. lateral

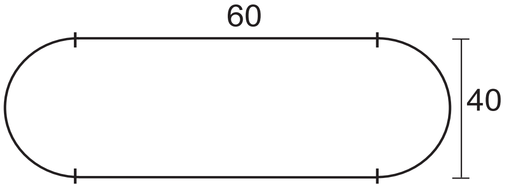
B. net

C. slant height

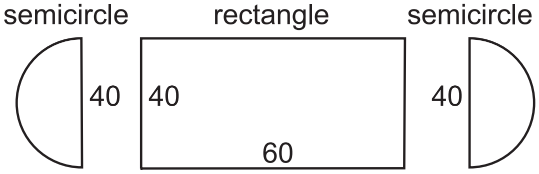
D. surface area (*S.A.*)

Problem Solving

How would you find the area of this odd shape?



If you look closely you can find a rectangle and 2 semicircles. If you put the semicircles together, you would, of course, have a full circle with a diameter of 40 feet. The rectangle has length 60 feet and width 40 feet.



To find the area of this odd shape you need to add the area of the circle and the area of the rectangle.

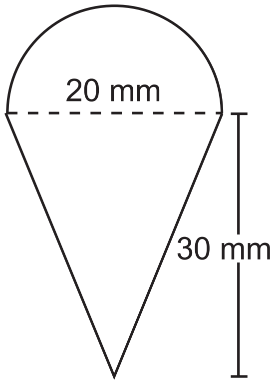
Remember: The radius is $\frac{1}{2}$ the diameter.

Area of Circle	+	**Area of Rectangle**	=	**Area of Total Shape**
$A = \pi r^2$		$A = lw$		
$(3.14)(20)^2$		$(60)(40)$		
1,256 ft^2	+	2,400 ft^2	≈	3,656 ft^2

Practice

Answer the following. Refer to **formulas** *in unit or in* **Appendix C** *as needed.*

1. Find the area of this shape.

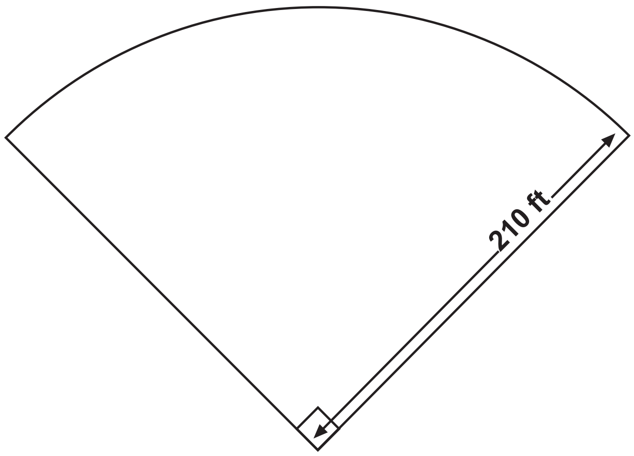


Hint: Do you see a semicircle and a triangle? Add their areas together.

Area of Semicircle	+	**Area of Triangle**	=	**Area of Total Shape**
$A = \frac{1}{2}(\pi r^2)$	+	$A = \frac{1}{2}bh$	$\approx$	

Hint: Your answer should be between 400 and 500 square millimeters (mm^2).

2. A softball field is constructed in the shape of $\frac{1}{4}$ of a circle, as shown in the diagram below:



If the coach has his players run the circular part of the field, what would be the total distance that a player would run?

Hint: Use the formula for circumference of a circle.

$C = \pi d$ *or*
$C = 2\pi r$

Remember that a player will be running only $\frac{1}{4}$ of the circle.

How many feet would a player run if he runs the circular part of the field 4 times? ____

Hint: Your answer should be between 1,000 and 1,500 feet.

There are 5,280 feet in a mile. Approximately how many times would you need to run just the circular part of the field to run a mile?

____

3. The coach in problem number 2 decides to have his players run around the entire field.

   Hint: circular part + 210 + 210

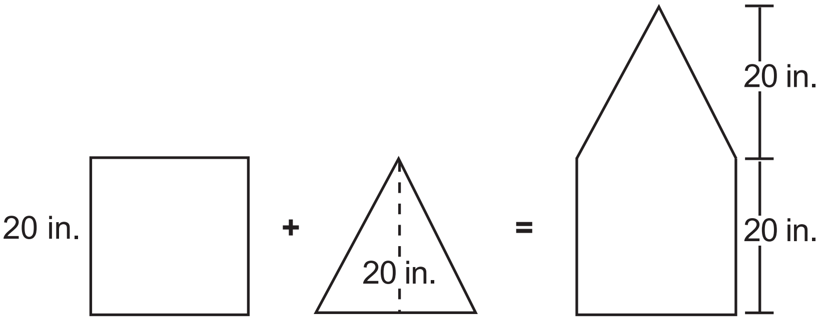
   What would be the total distance that a player would run? _____

   If the players run around the field 5 times:

   How many feet would this be? ______________________________

   Is this more than a mile or less than a mile? ___________________

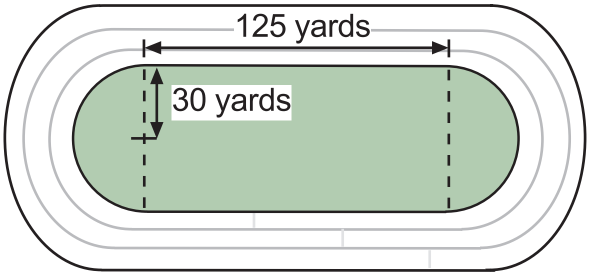
4. A square and a triangle can be combined to make an irregular pentagon.



   What would be the area of this figure? _______________________

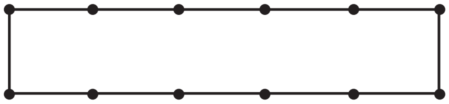
5. Susie is helping to replant the grass area inside the track at her high school. She needs to find the area of the shaded region shown in the diagram below so she can order the sod. The radius of each half-circle end of the inside of the track is 30 yards.

What is the area, in square yards (yd²), of the shaded region? ______



6. Ben has ordered 12 sections of fencing connected by hinges. Each section is 1 foot. Ben plans to make a pen for his dog, and he wants to give his dog the largest pen possible. Here is one possibility.

Find four other ways to build the pen. Fill in the chart below.



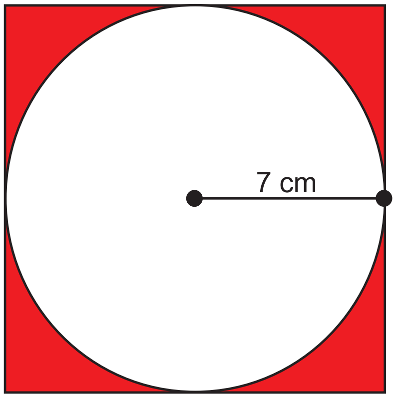
Ways to Build a Dog Pen

length	width	area (square feet—ft^2)
5	1	5

Which arrangement gives the maximum or largest area?________

What kind of rectangle is this? ______________________________

7. A circle with radius 7 cm is inscribed in the square below. Find the area of the shaded region.



Hint: To solve this problem, you have to use your imagination. If you cut the circle out with a pair of scissors, you would have the shaded part left!

area of square – area of circle = area of shaded area

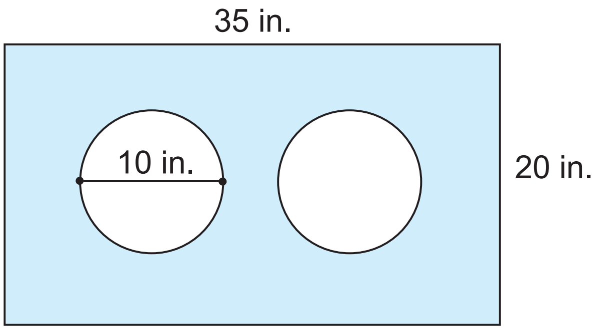
Find the answer. Be sure that you figure the length and width of the square correctly! The final answer should be between 40 and 50 square centimeters (cm^2).

____

8. Bob drills 2 holes with diameters of 10 inches from a board 35 inches by 20 inches.

Hint: rectangle – 2 circles = what is left

How much of the board is left? __________________________



Practice

Circle the letter of the correct answer. Refer to **formulas** *in unit or in* **Appendix C** *as needed.*

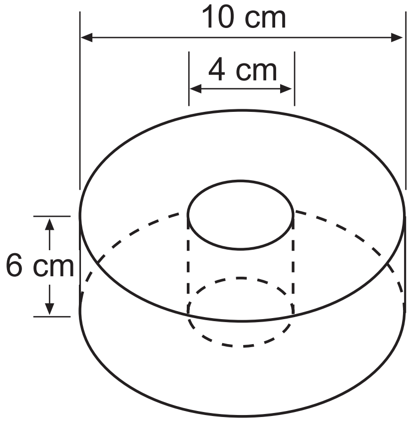
1. An engineer is designing a metal gasket for a machine. The gasket is in the shape of a cylinder with a cylindrical hole through its center. The diameter of the gasket is 10 centimeters, and its height is 6 centimeters. The diameter of the hole is 4 centimeters.

   Hint: volume of big cylinder – volume of little cylinder = volume of the gasket

   What is the volume of the gasket? ____________________________

   Your answer will be in cubic centimeters. The possible answers have π in them, so do not replace π with 3.14 in this problem. Make sure that you use the correct formula.

   a. 126π cubic centimeters
   b. 504π cubic centimeters
   c. 81π cubic centimeters
   d. 174π cubic centimeters

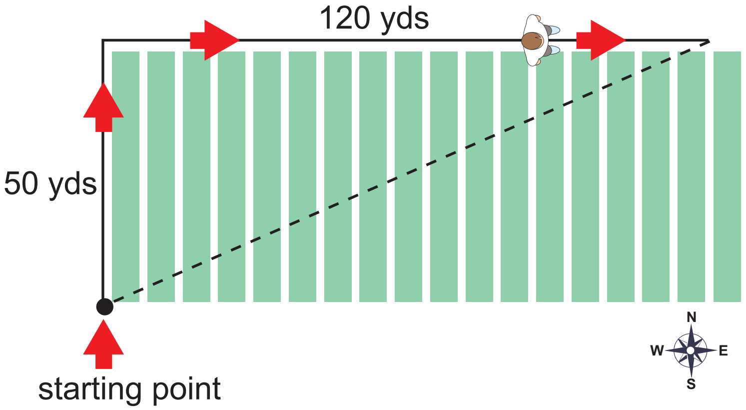


2. Wesley is pacing off a rectangular field. He walks 50 yards north and then 120 yards east. He suddenly gets thirsty and realizes that he has left his bottle of water back at the beginning point.

   Hint: Remember Pythagoras.

   How far is it from this point in a straight line to the starting point?

   ____



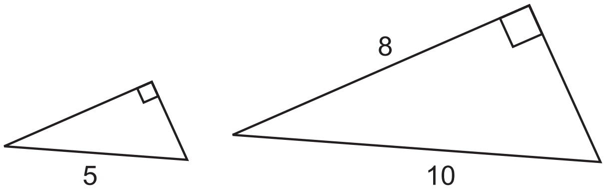
   a. 70 yards
   b. 170 yards
   c. 130 yards
   d. 125.4 yards

3. A small triangle and a large triangle are similar right triangles.

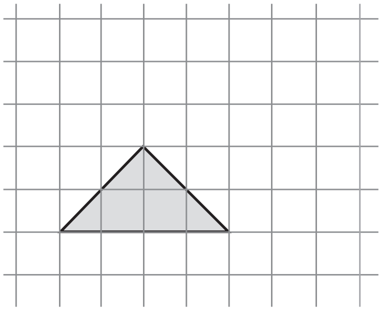
   Hint: Remember that you need to know the height of the small triangle.

   What is the area of the small triangle? ______________________

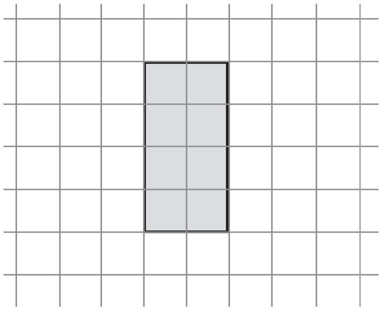
   a. 20 square feet
   b. 6 square feet
   c. 12 square feet
   d. 60 square feet



4. How would you compare the area of the triangle to the area of the rectangle?



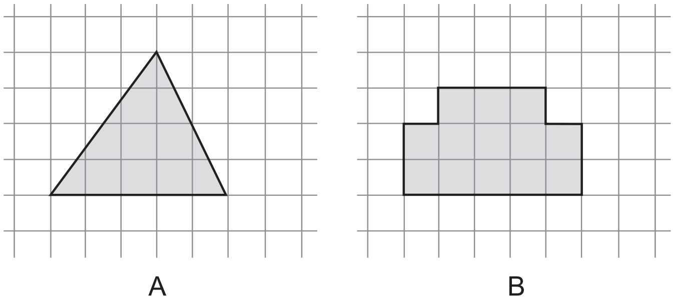
A



B

   a. They are equal in area.
   b. The area of the triangle is $\frac{1}{2}$ the area of the rectangle.
   c. The area of the triangle is twice the area of the rectangle.
   d. You can't compare the areas of different figures.

5. What is the ratio of the area of the triangle to the area of the irregular polygon?



a. 20:13
b. 10:13
c. 13:10
d. 7:13

Answer the following.

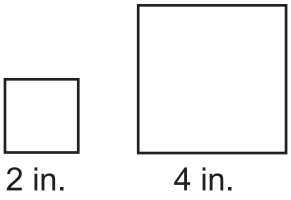
6. Consider the following two squares. Notice that the sides of the larger square are twice as long as the sides of the smaller square.

Find the ratio of the area of the large square to the area of the small square. ______________________________

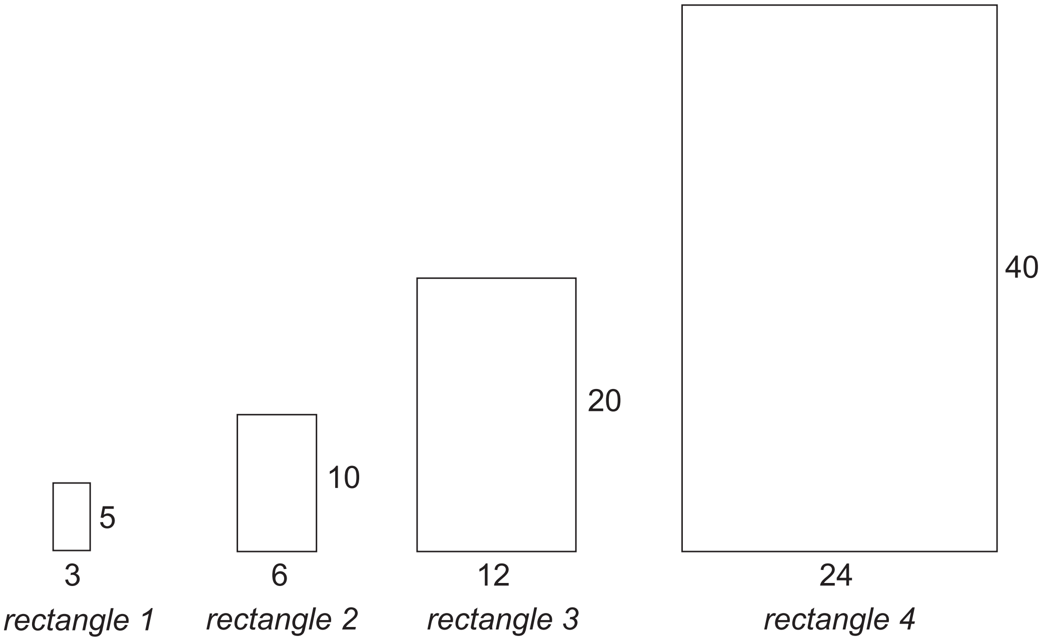
If we doubled the sides of the squares, do we double the areas?

____

By how much do the areas increase? ______________



7. Consider these four rectangles, and note how the dimensions have been changed.



Find the following ratios:

$\frac{\text{area of rectangle 4}}{\text{area of rectangle 1}}$ $\frac{\text{area of rectangle 3}}{\text{area of rectangle 1}}$ $\frac{\text{area of rectangle 2}}{\text{area of rectangle 1}}$

____________________ ____________________ ____________________

If we continue this pattern, what would be the ratio of the area of rectangle 5 to the area of rectangle 1? __________________________

8. Mr. Rodriguez is having some photos enlarged for his office. He wants to enlarge a photo that is 5 inches by 7 inches so the dimensions are *3 times larger* than the original.

   How many times larger than the area of the original photo will the area of the

   new photo be? ____

9. A circle has an area of 803.84 square meters.

   Hint: To find the radius, you must work backward. You know that to find the area of a circle we use this formula:

   Fill in what you know to figure out the radius.

$$\begin{aligned} A &= \pi r^2 \\ 803.84 &\approx (3.14)r^2 \\ \frac{803.84}{3.14} &\approx \frac{3.14}{3.14} r^2 \\ 256 &\approx r^2 \\ 16 &\approx r \end{aligned}$$

   divide both sides by 3.14

   If you know the radius, you can now use the circumference formula that uses the radius ($C = 2\pi r$) to find the final answer.

   What is the circumference of the circle?_____________________

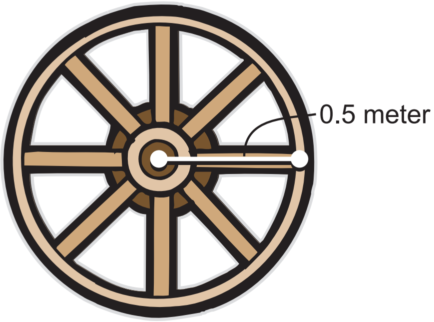
10. A sphere has a surface area of 1256 square meters. What is the radius of the sphere? Circle the letter of the correct answer.

    a. 10 meters
    b. 20 meters
    c. 100 meters
    d. 200 meters

11. On the wagon wheel below, it is 0.5 meter from the middle of the wheel to the outside of the wheel.

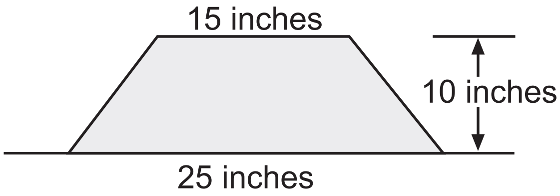
What is the circumference of the wheel? ______________________

How many meters will the wheel travel in 10 revolutions? ______



12. Pat is building a model railroad. He plans to build a stone bridge over an imaginary river. Each side of the bridge is shaped like a trapezoid. The dimensions of a side are given in the picture below. He needs to know the area so that he can buy enough rocks.

What is the total area of the 2 sides of the bridge? ______________

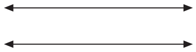


Practice

Match each definition with the correct term. Write each letter on the line provided.

______ 1. to meet or cross at one point

______ 2. two lines in the same plane that never meet

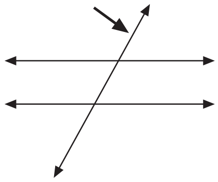


______ 3. figures or objects that are the same shape and the same size (≅)

______ 4. a portion of a line that begins at a point and goes on forever in one direction (→)

______ 5. a portion of a line that has a defined beginning and end (—)

______ 6. a line that intersects two or more other (usually parallel) lines in the same plane



______ 7. a straight line that is endless in length (↔)

A. congruent

B. intersect

C. line

D. line segment

E. parallel lines

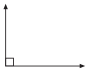
F. ray

G. transversal

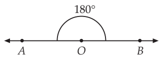
Practice

Match each graphic with the correct term. Write the letter on the line provided.

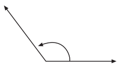
______ 1. acute angle

A. 

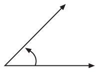
______ 2. obtuse angle

B. 

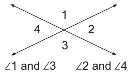
______ 3. right angle

C. 

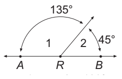
______ 4. straight angle

D. 

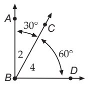
______ 5. complementary angles

A. 

______ 6. measure of an angle

B. 

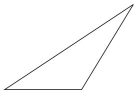
______ 7. supplementary angles

C. 

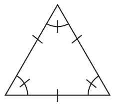
______ 8. vertical angles

D. m$\angle$

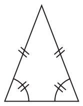
______ 9. equilateral triangle

A. 

______ 10. isosceles triangle

B. 

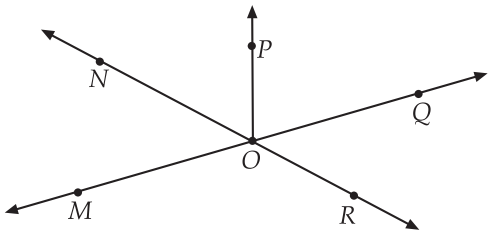
______ 11. obtuse triangle

C. 

Unit Review

Part A

Use the figure below to find the following.



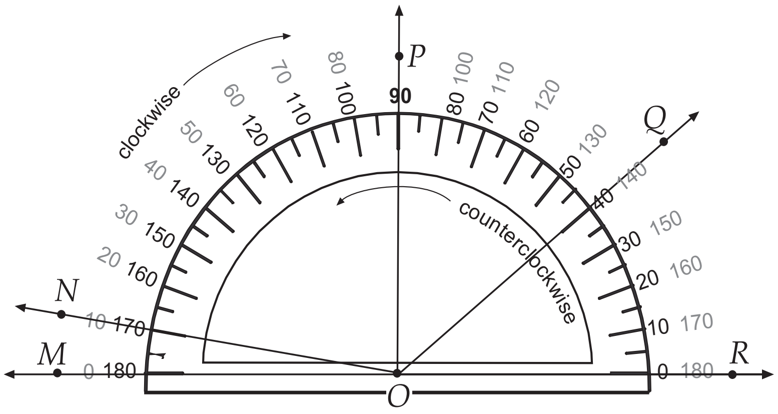
1. Write another name for $\overleftrightarrow{MQ}$. ______________________________

2. Name two lines that intersect. ______________________________

3. Are $\overrightarrow{OR}$ and $\overrightarrow{RO}$ opposite rays? ______________________________

4. Are $\overline{OR}$ and $\overline{RO}$ congruent? ______________________________

Use the **protractor** *below to find the* **measure of each angle**. *Then write whether the angle is* **acute**, **right**, *or* **obtuse**.



5. ∠RON ______________________________

6. ∠MON ______________________________

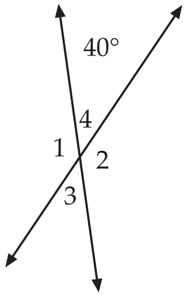
7. ∠POQ ______________________________

Use the **angles on the protractor** *on the previous page to identify the following.*

8. a right angle ______________________________

9. a straight angle ______________________________

Use the figure below to find the following.



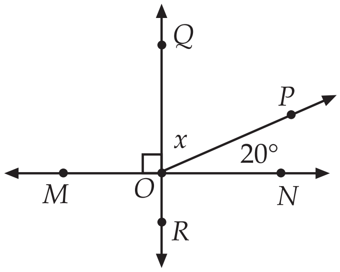
10. m∠1 ______________________________

11. m∠2 ______________________________

12. m∠3 ______________________________

13. Name two supplementary angles. ____________________

Use the figure below to find the following.

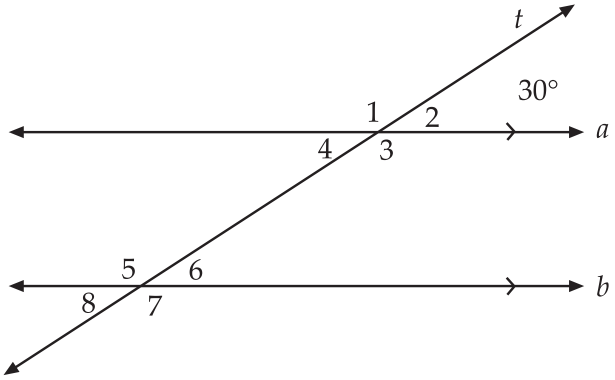


14. Find m∠x. ______________________________

15. Name a pair of complementary angles. ____________________

16. Name a pair of vertical angles. ____________________

Use the figure below to find the following.



17. List each angle whose measure is 30 degrees.__________________

18. Find m∠7. ____

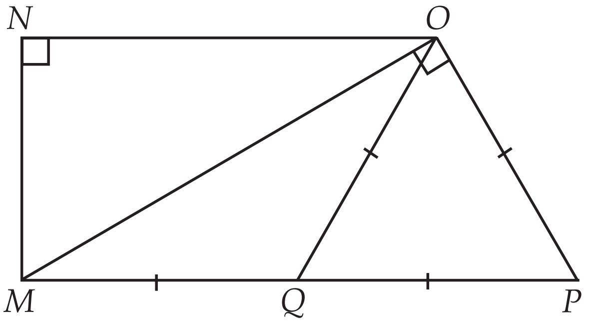
19. Name two parallel lines.____________________________________

20. Name the transversal. ______________________________________

21. ∠4 and ∠6 are _________________________ interior angles.

22. ∠2 and ∠6 are _________________________ angles.

Use the figure below to find the following.



23. Name two right angles. ________________________________

24. Name an isosceles triangle. _____________________________

25. Name an equilateral triangle ____________________________

26. Name an obtuse triangle. _______________________________

27. Find the m $\angle NMO$ if the m $\angle NOM$ is 30 degrees (°). __________

28. Find the length of MP if MQ is 10 inches. ___________________

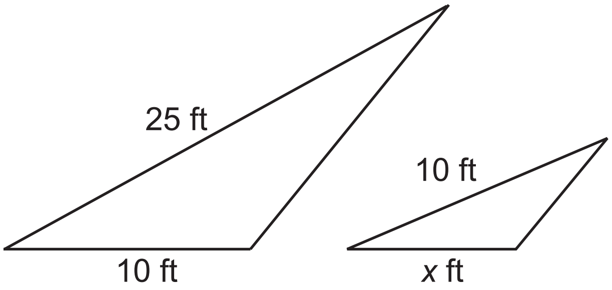
Unit Review

Part B

Numbers 29 and 30 are **gridded-response items**.
Write answers along the top of the grid and correctly mark them below.

The following shapes in numbers 29 and 30 are **similar**. *Set up a* **proportion** *to* **solve for** *x*.

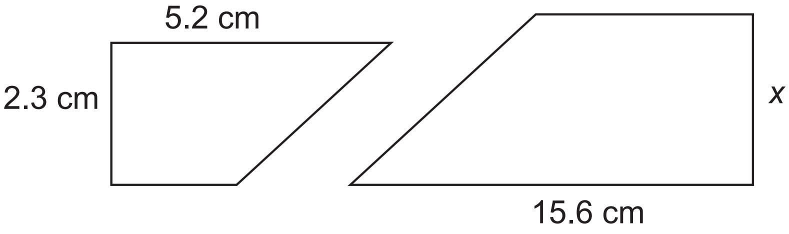
29.



Mark your answers on the grid to the right.

	⊘	⊘	⊘	
⊙	⊙	⊙	⊙	⊙
⓪	⓪	⓪	⓪	⓪
①	①	①	①	①
②	②	②	②	②
③	③	③	③	③
④	④	④	④	④
⑤	⑤	⑤	⑤	⑤
⑥	⑥	⑥	⑥	⑥
⑦	⑦	⑦	⑦	⑦
⑧	⑧	⑧	⑧	⑧
⑨	⑨	⑨	⑨	⑨

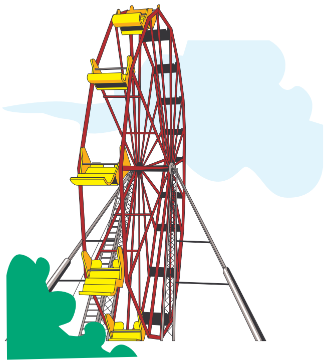
30.



Mark your answers on the grid to the right.

	⊘	⊘	⊘	
⊙	⊙	⊙	⊙	⊙
⓪	⓪	⓪	⓪	⓪
①	①	①	①	①
②	②	②	②	②
③	③	③	③	③
④	④	④	④	④
⑤	⑤	⑤	⑤	⑤
⑥	⑥	⑥	⑥	⑥
⑦	⑦	⑦	⑦	⑦
⑧	⑧	⑧	⑧	⑧
⑨	⑨	⑨	⑨	⑨

31. When a Ferris wheel casts a 20-meter shadow, a post 1.6 meters tall casts a 2.4-meter shadow.



First estimate based on what you know, then set up a proportion and solve.

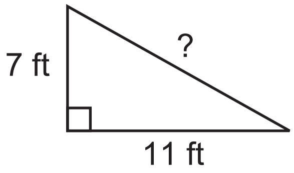
How tall is the Ferris wheel? ____________
Round to nearest tenth.

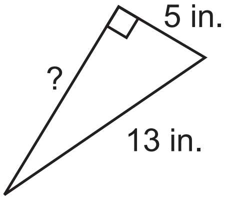
Is your answer reasonable? Explain. ______________________________

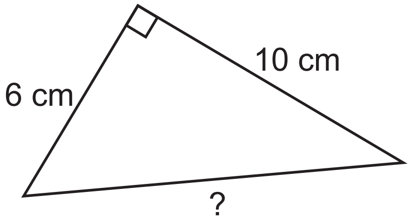
____

____

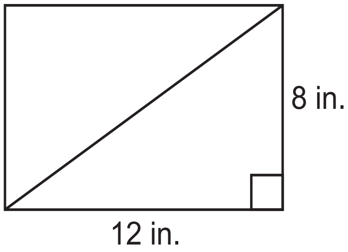
Using the **Pythagorean theorem $a^2 + b^2 = c^2$** *to find the* **length of the missing side**. **Round to nearest hundredth.**

32. 

33. 

34. 

35. Find the diagonal of this rectangle. ______________________________

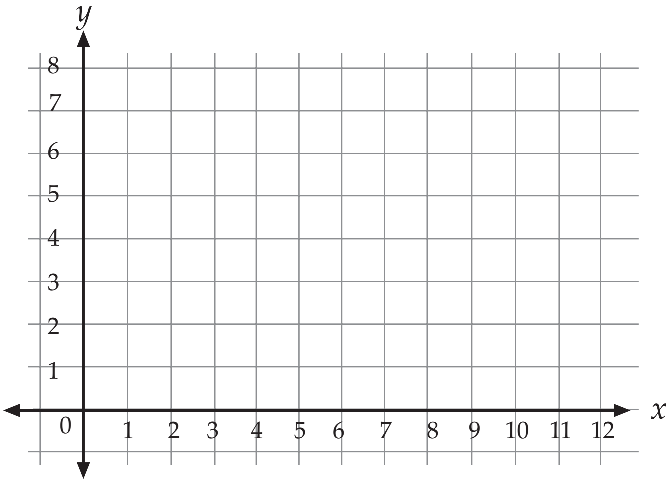


Complete the following.

36. Use the coordinate grid below to plot these points:
(1, 1), (9, 1), (9, 4), (1, 4)

Connect the points to form a rectangle.

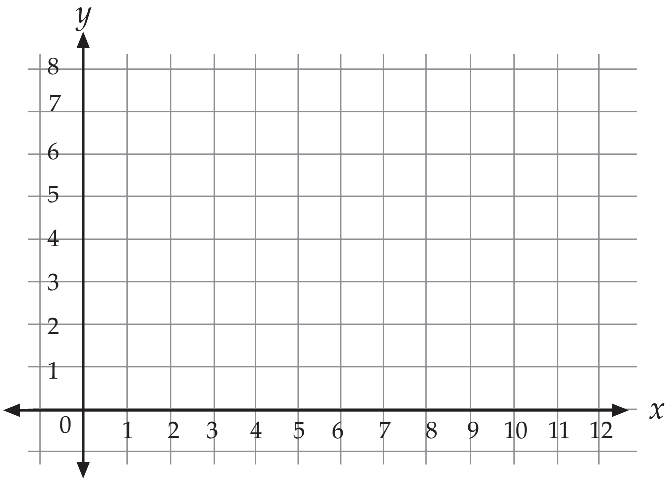
What is the area of the rectangle? ______________________________



37. Use the coordinate grid below to plot these points.
(3, 1), (8, 1), (1, 4)

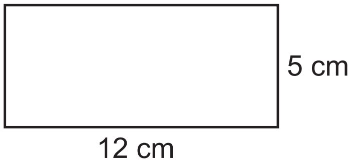
Connect the points to form a triangle.

What is the area of the triangle? ______________________________

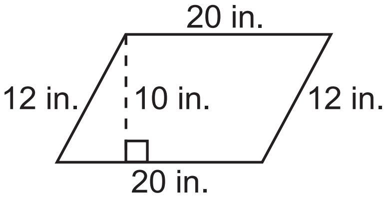


Use the **reference sheet** *in* **Appendix C** *to find the appropriate* **formulas**. *Then find the* **area** *of the following figures.*

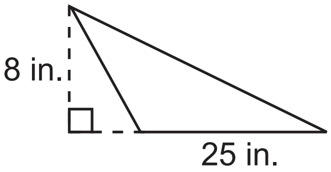
38.



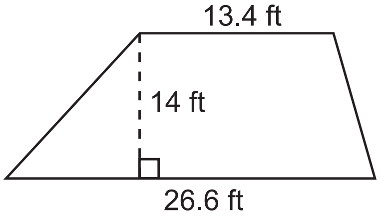
39.



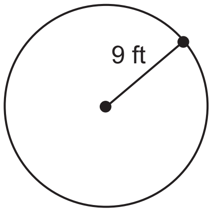
40.



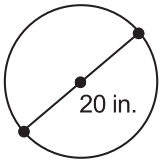
41.



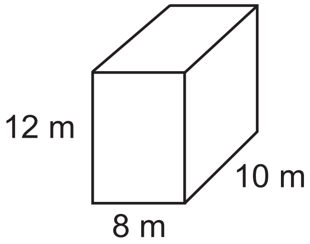
42.



43.

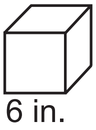


Use the **reference sheet** *in* **Appendix C** *to find the appropriate* **formulas**. *Then find the* **volume** *and* **surface area** *of the following figures.*



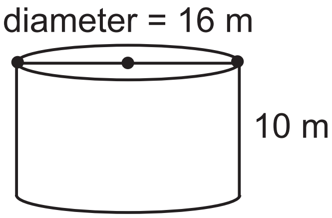
44. volume ____

45. surface area ____



46. volume ____

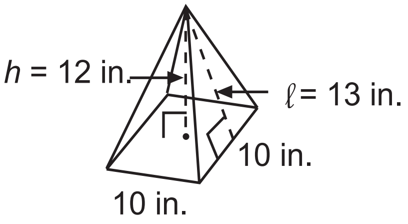
47. surface area ____



48. volume ____

49. surface area ____

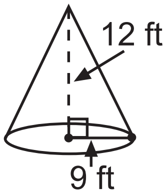
50.



volume ____________

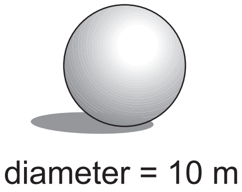
For the following two problems, leave your answer in π **form**.

51.



volume ____________

52.



volume ____________

Bonus Problems:

53. Refer to the picture on problem 50. Find the surface area if the slant height (ℓ) is 13 inches.

54. A circle has area 200.96 square inches. First find the radius and then find the circle's circumference.

Unit 5: Data Analysis and Probability

This unit emphasizes how statistical methods, probability concepts, and measures of central tendencies are used to collect and interpret data to solve problems.

Unit Focus

Number Sense, Concepts, and Operations

- Understand and use the real number system. (MA.A.2.4.2)
- Understand and explain the effects of addition, subtraction, multiplication, and division on real numbers. (MA.A.3.4.1)
- Apply special number relationships such as sequences to real-world problems. (MA.A.5.4.1)

Algebraic Thinking

- Describe, analyze, and generalize relationships, patterns, and functions using words, symbols, variables, tables, and graphs. (MA.D.1.4.1)
- Determine the impact when changing parameters of given functions. (MA.D.1.4.2)
- Represent real-world problem situations using finite graphs. (MA.D.2.4.1)
- Use systems of equations and inequalities to solve real-world problems graphically and algebraically. (MA.D.2.4.2)

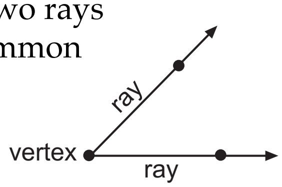
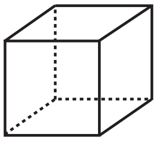
Data Analysis and Probability

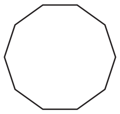
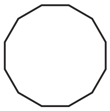
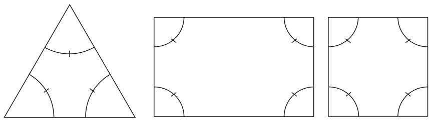
- Interpret data that has been collected, organized, and displayed in charts, tables, and plots. (MA.E.1.4.1)

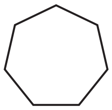
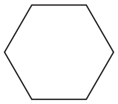
- Calculate measures of central tendency (mean, median, and mode) and dispersion (range) for complex sets of data and determine the most meaningful measure to describe the data. (MA.E.1.4.2)
- Analyze real-world data and make predictions of larger populations by using the sample population data and using appropriate technology, including calculators and computers. (MA.E.1.4.3)
- Determine probabilities using counting procedures, tables, and tree diagrams. (MA.E.2.4.1)
- Determine the probability for simple and compound events as well as independent and dependent events. (MA.E.2.4.2)
- Explain the limitations of using statistical techniques and data in making inferences and valid arguments. (MA.E.3.4.2)

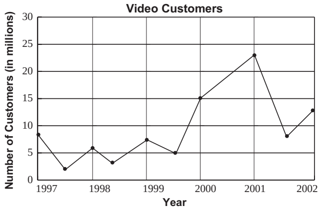
Vocabulary

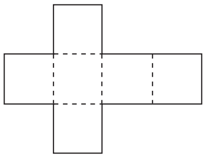
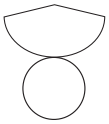
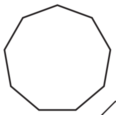
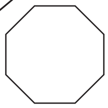
Use the vocabulary words and definitions below as a reference for this unit.

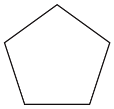
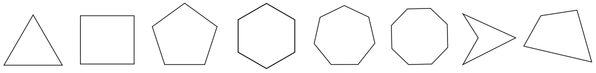
angle (∠)	the shape made by two rays extending from a common endpoint, the vertex; measures of angles are described in degrees (°) 
congruent (≅)	figures or objects that are the same shape and the same size
coordinate grid or system	network of evenly spaced, parallel horizontal and vertical lines especially designed for locating points, displaying data, or drawing maps
coordinate plane	the plane containing the x- and y-axes
coordinates	numbers that correspond to points on a graph in the form (x, y)
cube	a rectangular prism that has six square faces 
data	information in the form of numbers gathered for statistical purposes

data display	different ways of displaying data in tables, charts, or graphs *Example*: pictographs; circle graphs; single, double, or triple bar and line graphs; histograms; stem-and-leaf plots; and scatterplots
decagon	a polygon with 10 sides 
degree (°)	common unit used in measuring angles
dependent events	two events in which the first affects the outcome of the second event
difference	the result of a subtraction *Example*: In 16 – 9 = 7, 7 is the difference.
dodecagon	a polygon with 12 sides 
equally likely	two or more possible outcomes of a given situation that have the same probability
equation	a mathematical sentence that equates one expression to another expression *Example*: $2x = 10$
equiangular polygon	a polygon with all angles equal 
even number	any whole number divisible by 2 *Example*: 2, 4, 6, 8, 10, 12 …

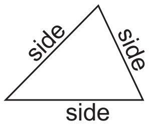
event	a possible result or outcome in probability
factor	a number or expression that divides exactly another number *Example*: 1, 2, 4, 5, 10, and 20 are factors of 20.
graph	a drawing used to represent data *Example*: bar graphs, double bar graphs, circle graphs, and line graphs
graph of an equation	all points whose coordinates are solutions of an equation
heptagon	a polygon with seven sides 
hexagon	a polygon with six sides 
independent events	two events in which the outcome of the first event does *not* affect the outcome of the second event
inequality	a sentence that states one expression is greater than (>), greater than or equal to (≥), less than (<), less than or equal to (≤), or not equal to (≠) another expression *Example*: $a \neq 5$ or $x < 7$
infinite	having no boundaries or limits
line (↔)	a straight line that is endless in length 

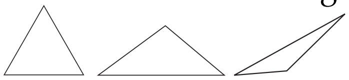
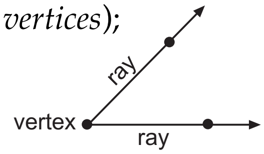
linear equation	an equation whose graph in a coordinate plane is a straight line; an algebraic equation in which the variable quantity or quantities are in the first power only and the graph is a straight line *Example*: $20 = 2(w + 4) + 2w$; $y = 3x + 4$
line graph	a graph used to show change over time in which line segments are used to indicate amount and direction 
line of best fit (on a scatterplot)	line drawn as near as possible to the various points so as to best represent the trend being graphed; also called a *trend line*
mean (or average)	the arithmetic average of a set of numbers
measure (m) of an angle (∠)	the number of degrees (°) of an angle
measures of central tendency	the mean, median, and mode of a set of data
median	the middle point of a set of ordered numbers where half of the numbers are above the median and half are below it
mode	the score or data point found most often in a set of numbers

multiples	the numbers that result from multiplying a given number by the set of whole numbers *Example*: The multiples of 15 are 0, 15, 30, 45, 60, 75, etc.
net	a plan which can be used to make a model of a solid; a two-dimensional shape that can be folded into a three-dimensional figure  net of a cube  net of cone
nonagon	a polygon with nine sides 
octagon	a polygon with eight sides 
odd number	any whole number *not* divisible by 2 *Example*: 1, 3, 5, 7, 9, 11 …
ordered pair	the location of a single point on a rectangular coordinate system where the digits represent the position relative to the x-axis and y-axis *Example*: (x, y) or (3, 4)
outcome	a possible result of a probability experiment

pattern (relationship)	a predictable or prescribed sequence of numbers, objects, etc.; also called a *relation* or *relationship*; may be described or presented using manipulatives, tables, graphics (pictures or drawings), or algebraic rules (functions) *Example*: 2, 5, 8, 11 … is a pattern. Each number in this sequence is three more than the preceding number. Any number in this sequence can be described by the algebraic rule, $3n - 1$, by using the set of counting numbers for n.
pentagon	a polygon with five sides 
plane	an undefined, two-dimensional (no depth) geometric surface that has no boundaries specified; a flat surface
point	a location in space that has no length or width
polygon	a closed plane figure whose sides are straight lines and do not cross *Example*: triangle (3 sides), quadrilateral (4 sides), pentagon (5 sides), hexagon (6 sides), heptagon (7 sides), octagon (8 sides); concave, convex 

prime number	any whole number with only two factors, 1 and itself *Example*: 2, 3, 5, 7, 11, etc.
probability	the ratio of the number of favorable outcomes to the total number of outcomes
quadrilateral	polygon with four sides *Example*: square, parallelogram, trapezoid, rectangle, rhombus, concave quadrilateral, convex quadrilateral 
range (of a set of numbers)	the difference between the highest (H) and the lowest value (L) in a set of data; sometimes calculated as H – L + 1
ratio	the quotient of two numbers used to compare two quantities *Example*: The ratio of 3 to 4 is $\frac{3}{4}$.
scatterplot (or scattergram)	a graph of data points, usually from an experiment, that is used to observe the relationship between two variables
sequence	an ordered list with either a constant difference (arithmetic) or a constant ratio (geometric)
set	a collection of distinct objects or numbers

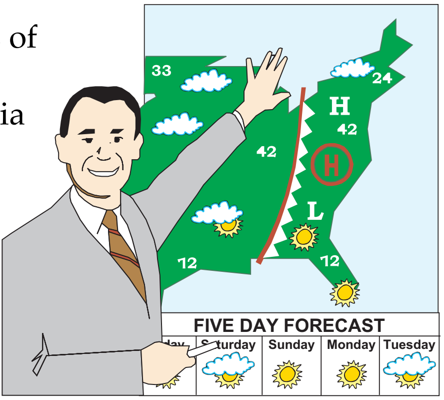
side	the edge of a two-dimensional geometric figure *Example*: A triangle has three sides. 
simplest form	a fraction whose numerator and denominator have no common factor greater than 1 *Example*: The simplest form of $\frac{3}{6}$ is $\frac{1}{2}$.
slope	the steepness of a line, defined by the ratio of the change in y to the change in x
solution	any value for a variable that makes an equation or inequality a true statement *Example*: In $y = 8 + 9$ $y = 17$ 17 is the solution.
substitute	to replace a variable with a numeral *Example*: $8(a) + 3$ $8(5) + 3$
sum	the result of an addition *Example*: In $6 + 8 = 14$, 14 is the sum.
table (or chart)	an orderly display of numerical information in rows and columns
tree diagram	a diagram in which all the possible outcomes of a given event are displayed

triangle	a polygon with three sides; the sum of the measures of the angles is 180° 
value (of a variable)	any of the numbers represented by the variable
variable	any symbol that could represent a number
vertex	the common endpoint from which two rays begin or the point where two lines intersect; the point on a triangle or pyramid opposite to and farthest from the base; (plural: *vertices*); vertices are named clockwise or counterclockwise 
whole number	any number in the set {0, 1, 2, 3, 4, …}
***x*-axis**	the horizontal (↔) axis on a coordinate plane
***y*-axis**	the vertical (↕) axis on a coordinate plane

Unit 5: Data Analysis and Probability

Introduction

In analyzing data, we learn to organize sets of statistics in ways that enable us to be better decision makers. The various forms of media which provide us with information frequently use graphs, measures of central tendency (mean, median, and mode), and probability. For example, television weather reporters can predict for a week at a time the probability of rain for a given geographic area.



Therefore, in order to keep up with current events, a person needs to be able to formulate hypotheses, collect and interpret data, and draw conclusions based on statistics, tables, graphs, and charts. Furthermore, we need to be able to recognize ways in which statistics can be misleading. Clever statisticians can devise graphs, charts, etc., that are deceptive. With a good knowledge of data analysis, people can develop skills for interpreting and evaluating statistical presentations.

Lesson One Purpose

- Describe, analyze, and generalize relationships, patterns, and functions using words, symbols, variables, tables, and graphs. (MA.D.1.4.1)
- Determine the impact when changing parameters of given functions. (MA.D.1.4.2)
- Represent real-world problem situations using finite graphs. (MA.D.2.4.1)
- Use systems of equations and inequalities to solve real-world problems graphically and algebraically. (MA.D.2.4.2)

- Interpret data that has been collected, organized, and displayed in charts, tables, and plots. (MA.E.1.4.1)
- Analyze real-world data and make predictions of larger populations by using the sample population data and using appropriate technology, including calculators and computers. (MA.E.1.4.3)

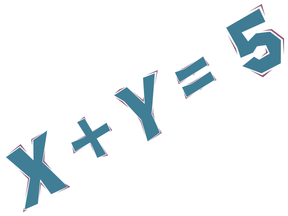
Equations in Two Variables

In this unit, we will study **equations** with two **variables**. We will limit our study to equations such as these:

$2x + y = 10$; $y = 3x + 5$; and $y = 4x$.

To begin, let's see how we can solve an equation like

$x + y = 5$.



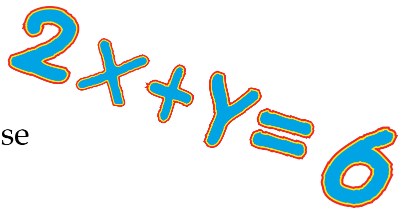
You will find that, unlike the earlier equations in Unit 3, this type of equation has an **infinite** number (*no limit* to the number) of **solutions**. A *solution of an equation with two variables* is an **ordered pair** of numbers that make the equation true.

Suppose that we replace the variable x with the **value** of 0. Obviously, the variable y would have to be replaced with the number 5, because

$0 + 5 = 5$.

So if $x = 0$, then $y = 5$, and we use the *ordered pair* (0, 5) to denote the solution.

Here are two other solutions for $x + y = 5$.



- If we let $x = 1$, then y would be 4 because

  $1 + 4 = 5$.

Hence, (1, 4) is a solution.

- If we let $x = 2$, then y would be 3, because

  $2 + 3 = 5$.

Therefore, (2, 3) is also a solution.

Many ordered pairs work for the equation $x + y = 5$.

Below is a **table** *of values* with several examples of ordered pairs that are solutions to the equation. A *table* is an orderly display of numerical information in rows and columns.

$x + y = 5$	
0	5
1	4
2	3
3	2
1	4
5	0
$\frac{1}{2}$	$4\frac{1}{2}$
0.25	4.75
⋮	⋮

Example 1: Find five solutions for the equation $y = x - 2$.

Solution: Begin by picking *values* for x. Any number will do.

Equation	x	Substitute for x	Solve for y	Solution
$y = x - 2$	0	$y = \mathbf{0} - 2$	-2	(0, -2)
	1	$y = \mathbf{1} - 2$	-1	(1, -1)
	2	$y = \mathbf{2} - 2$	0	(2, 0)
	-1	$y = \mathbf{-1} - 2$	-3	(-1, -3)
	-5	$y = \mathbf{-5} - 2$	-7	(-5, -7)

Remember: In an ordered pair, the value of x is listed first, and the value for y is listed second—even if the variable y is written first in the equation.

Example: In the equation $y = x - 2$, the solution is (0, -2).

$$\underset{\substack{\downarrow \\ -2}}{y} = \underset{\substack{\downarrow \\ 0}}{x} - 2 \qquad (\underset{\substack{\downarrow \\ x}}{0}, \underset{\substack{\downarrow \\ y}}{-2})$$

$-2 = 0 - 2$

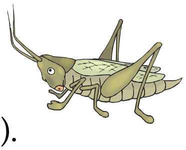
$-2 = -2$

Example 2: Find three solutions for the equation $2x + y = 6$.

Solution: Pick three values for x. Again, any number will do.

Equation	***x***	**Substitute for *x***	**Solve for *y***	**Solution**
$2x + y = 6$	0	$2\mathbf{(0)} + y = 6$	6	(0, 6)
	1	$2\mathbf{(1)} + y = 6$	4	(1, 4)
	-1	$2\mathbf{(-1)} + y = 6$	8	(-1, 8)

Example 3: $T = \frac{1}{4}C + 40$ shows the relationship between the number of times a cricket chirps per minute (C) and the temperature in degrees Fahrenheit (T). Solve this equation by finding three solutions.

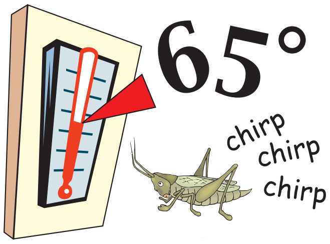


Solution: Pick any values for x. Numbers divisible by four would be the easiest to work with.

Equation	***T***	**Substitute for *C***	**Solve for *T***	**Solution**
$T = \frac{1}{4}C + 40$	4	$T = \frac{1}{4}(4) + 40$	41	(4, 41)
	16	$T = \frac{1}{4}(16) + 40$	44	(16, 44)
	40	$T = \frac{1}{4}(40) + 40$	50	(40, 50)

We just found that when a cricket chirps 40 times, then it is 50 degrees Fahrenheit.

Example 4: Using the information in example 3 on the previous page, is it true that when the temperature is 65 degrees Fahrenheit that a cricket will chirp 60 times?



Solution: We basically need to check to see if the ordered pair (60, 65) is a solution for the equation $T = \frac{1}{4}C + 40$.

$$\begin{array}{lrll} & T = & \frac{1}{4}C + 40 & \\ \text{Substitute} & 65 = & \frac{1}{4}(60) + 40 & \\ & 65 = & 15 + 40 & \\ & 65 = & \quad 55 & \qquad \text{This is } \textit{not} \text{ a true statement.} \end{array}$$

We now know that when the temperature is 65 degrees, a cricket will *not* chirp 60 times. Can you see that the cricket will only chirp 60 times when the temperature is 55 degrees Fahrenheit?

$$\begin{array}{lrll} & T = & \frac{1}{4}C + 40 & \\ \text{Substitute} & 55 = & \frac{1}{4}(60) + 40 & \\ & 55 = & 15 + 40 & \\ & 55 = & \quad 55 & \qquad \text{This is a true statement.} \end{array}$$

Example 5: Which ordered pair (-8, -5) or (12, -10) is a solution for the equation $y = \frac{3}{4}x + 1$?

Solution: Substitute the values for x and y into the equation:

(-8, -5)

$y = \frac{3}{4}x + 1$

Substitute $-5 = \frac{3}{4}(-8) + 1$

$-5 = -6 + 1$

$-5 = -5$ This is true, so (-8, -5) is a solution.

(12, -10)

$y = \frac{3}{4}x + 1$

Substitute $-10 = \frac{3}{4}(12) + 1$

$-10 = 9 + 1$

$-10 = 10$ This is false, so (12, -10) is not a solution.

Practice

Circle the letter of the correct answer.

1. Which ordered pair is a solution of $y = 3x - 4$?

   a. (-4, 0)
   b. (1, 1)
   c. (0, -4)
   d. (2, -2)

Complete the **table** *below for each equation.*

Equation	**x**	**Substitute for x**	**Solve for y**	**Solution**
2. $y = 2x - 3$	0	$y = 2(0) - 3$	________	________
	1	$y = 2(1) - 3$	________	________
	-4	$y = 2(-4) - 3$	________	________
3. $y = \frac{2}{5}x + 1$	5	$y = \frac{2}{5}(5) + 1$	________	________
	10	$y = \frac{2}{5}(10) + 1$	________	________
	0	$y = \frac{2}{5}(0) + 1$	________	________
4. $2x + y = 4$	0	$2(0) + y = 4$	________	________
	-1	$2(-1) + y = 4$	________	________
	5	$2(5) + y = 4$	________	________

Translate each sentence into an **equation***.*

5. The sum of two numbers (x and y) is 15. ____________________

6. Twice the first number (x) is 4 times the second number (y).

   ____

7. The first number (x) is 5 more than the second number (y).

____

8. The first number (x) is 3 less than the second number (y).

____

Find **three solutions** *for the following equations. You may* **let x be any value** *you wish.*

9. $x + y = 0$ __________ , __________ , __________

10. $y = 2x - 1$ __________ , __________ , __________

11. $y = \frac{1}{4}x + 1$

    Hint: Choose values for x that are divisible by 4.

    __________ , __________ , __________

Answer the following.

12. The Alachua County Water Department charges a monthly administrative fee of \$10.50, and \$0.0049 for each gallon of water used up to 5,000 gallons. What will be the monthly charge for a customer who used 4,500 gallons of water in one month? **Round to the nearest penny.**

Let C = total cost *and*
g = number of gallons used by the customer

The equation to figure the total cost, C, for use of up to 5,000 gallons is

$$C = 10.50 + 0.0049g$$

Practice

Match each definition with the correct term. Write the letter on the line provided.

______ 1. any value for a variable that makes an equation or inequality a true statement

______ 2. the location of a single point on a rectangular coordinate system where the digits represent the position relative to the *x*-axis and *y*-axis

______ 3. any of the numbers represented by the variable

______ 4. a mathematical sentence that equates one expression to another expression

______ 5. any symbol that could represent a number

______ 6. an orderly display of numerical information in rows and columns

______ 7. having no boundaries or limits

A. equation

B. infinite

C. ordered pair

D. solution

E. table
(or chart)

F. value
(of a variable)

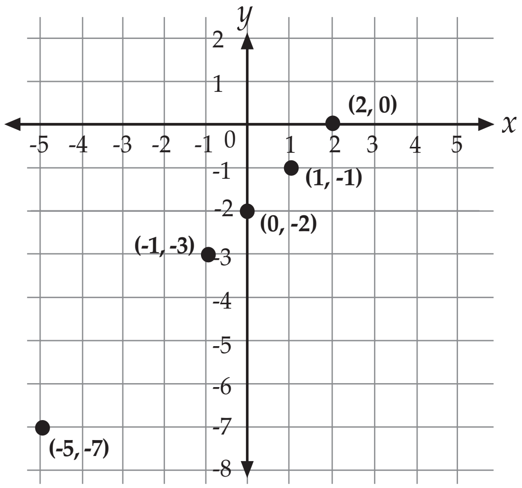
G. variable

Graphing Linear Equations

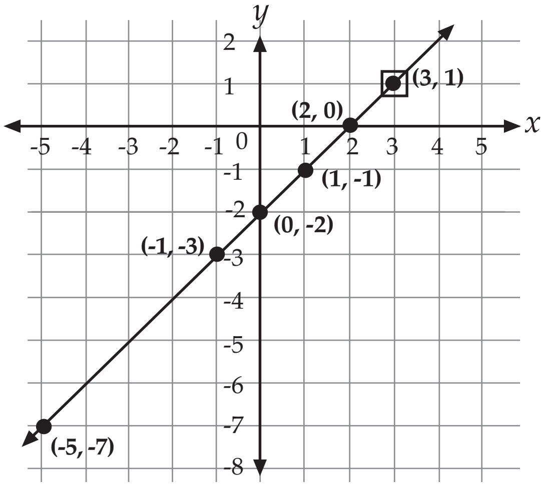
If you refer to example 1 in the last section, you will see that we found five solutions for the equation $y = x - 2$. Here is a *table of values* with the summary of the results.

x	y	
0	-2	(0, -2)
1	-1	(1, -1)
2	0	(2, 0)
-1	-3	(-1, -3)
-5	-7	(-5, -7)

The **graph of an equation** with two variables is all the **points** whose **coordinates** are solutions of the equation. The *coordinates* correspond to points on a **graph**. Let's graph these *points* and see what we get.



Carefully draw the line that connects these points.



The graph of the equation $y = x - 2$ is the **line (↔)** drawn on the **coordinate plane**. The **coordinate grid** itself contains the ***x*-axis**—the horizontal (↔) axis and ***y*-axis**—the vertical (↕) axis. The *line* drawn is endless in length and the **plane** is a flat surface with no boundaries.

Notice that the point (3, 1) lies on our line, but it was *not* one of our original points. Let's see if it is a solution. **Substitute** or *replace* the variables x and y with (3, 1).

$y = x - 2$	*substitute* 3 for x and 1 for y
$1 = 3 - 2$	
$1 = 1$	This is true, so (3, 1) is a solution of the equation.

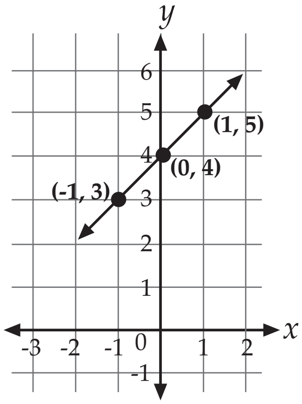
It turns out that any point on the line is a solution of the equation. We cannot write all the solutions to an equation because x can be anything, but we can draw a *picture* of the solutions using the *coordinate plane* and a line. The equations that we have been working with in the last section are called **linear equations** because their graphs are always *straight lines*.

Example 1: Graph $y = x + 4$.

Solution: It only takes two points to determine a line, so we only need two solutions of the equation. Generally, we find three solutions. (If we make an arithmetic mistake, it will be obvious because we will not get a line.) Choose three small values for x. Remember x can be any number that you wish.

Equation	*x*	Substitute for *x*	Solve for *y*	Solution
$y = x + 4$	0	$y = 0 + 4$	4	(0, 4)
	1	$y = 1 + 4$	5	(1, 5)
	-1	$y = -1 + 4$	3	(-1, 3)

Plot these points and draw the line.



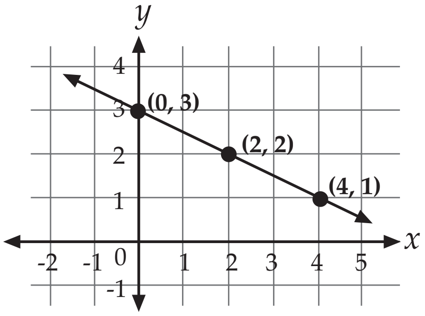
Remember: Any point on the line is a solution.

Example 2: Graph $y = -\frac{1}{2}x + 3$.

Solution: Choose values for x that are divisible by 2.

Equation	***x***	**Substitute for *x***	**Solve for *y***	**Solution**
$y = -\frac{1}{2}x + 3$	0	$y = -\frac{1}{2}(0) + 3$	3	(0, 3)
	2	$y = -\frac{1}{2}(2) + 3$	2	(2, 2)
	4	$y = -\frac{1}{2}(4) + 3$	1	(4, 1)

Plot these points and draw the line.



Remember: Any point on the line is a solution.

Practice

Graph the **linear equations** *below. For each equation in numbers 1-5, do the following steps:*

Step 1: Make a table of values.

Step 2: Find at least three ordered pairs that solve the equation.

Step 3: Graph each solution as a point on the coordinate plane on the following page.

Step 4: Connect the points with a straight line.

Step 5: Label the line with the equation.

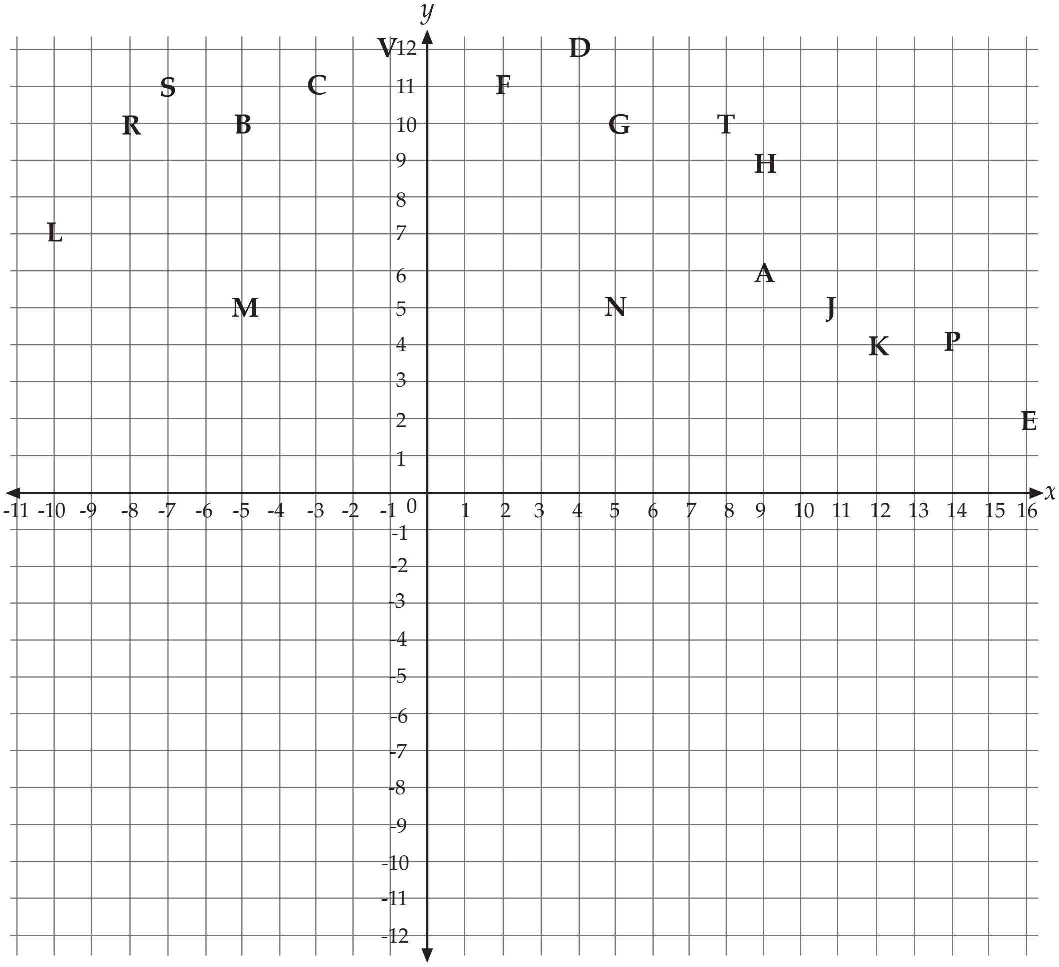
1. $y = 2x$

2. $y = \text{-}1x + 2$

3. $y = \frac{1}{4}x - 2$

4. $x - y = 3$

5. $y = x + 2$



✓ **Check yourself:** Next to each equation, write down the letter its line went through on the coordinate plane. Copy the letters on the line below. The letters will tell you how well you did on the assignment.

letters: ______________________________

Practice

Read the following.

What's my equation?

In this problem, we'll work backwards starting with the following ordered pairs:

(0, 0), (1, 2), (2, 4), (3, 6), and (4, 8).

List the ordered pairs vertically as you would to make a table of values. This will help you visually compare all the x and y values.

x	y
0	0
1	2
2	4
3	6
4	8

Think: What do I have to do to x to get y?

Answer: To get y, I have to double x. Therefore, the relationship between x and y expressed as an equation is $y = 2x$.

Let's try another one.

Find the equation that relates x and y using the following ordered pairs:

(1, 4), (2, 5), (3, 6), and (4, 7).

List ordered pairs vertically.

x	y
1	4
2	5
3	6
4	7

Think: What do I have to do to x to get y?

Answer: To get y, I have to add 3. Therefore, the relationship between x and y expressed as an equation is $y = x + 3$.

List the **ordered pairs vertically** *like the examples on the previous page.* **The first one is done for you.** *Then find the* **equation** *that relates* ***x*** *and* ***y****.*

1. (1, 1), (2, 2), (3, 3), (4, 4)

x	y
1	1
2	2
3	3
4	4

Equation: ____

2. (0, -2), (1, -1), (2, 0), (3, 1)

Equation: ____

3. (1, 2), (2, 1), (3, 0), (4, -1)

Equation: ____

4. (0, 5), (1, 6), (2, 7), (3, 8)

Equation: ____

5. (0, -1), (1, 1), (2, 3), (3, 5)

Equation: ____

Practice

Use the list below to write the correct term for each definition on the line provided.

coordinate grid
coordinate plane
coordinates
graph
graph of an equation
linear equation
plane
point
x-axis
y-axis

______________________ 1. an equation whose graph in a coordinate plane is a straight line

______________________ 2. a drawing used to represent data

______________________ 3. the plane containing the x- and y-axes

______________________ 4. the horizontal (↔) axis on a coordinate plane

______________________ 5. all points whose coordinates are solutions of an equation

______________________ 6. numbers that correspond to points on a graph in the form (x, y)

______________________ 7. a location in space that has no length or width

______________________ 8. the vertical (↕) axis on a coordinate plane

______________________ 9. network of evenly spaced, parallel horizontal and vertical lines especially designed for locating points, displaying data, or drawing maps

______________________ 10. an undefined, two-dimensional (no depth) geometric surface that has no boundaries specified; a flat surface

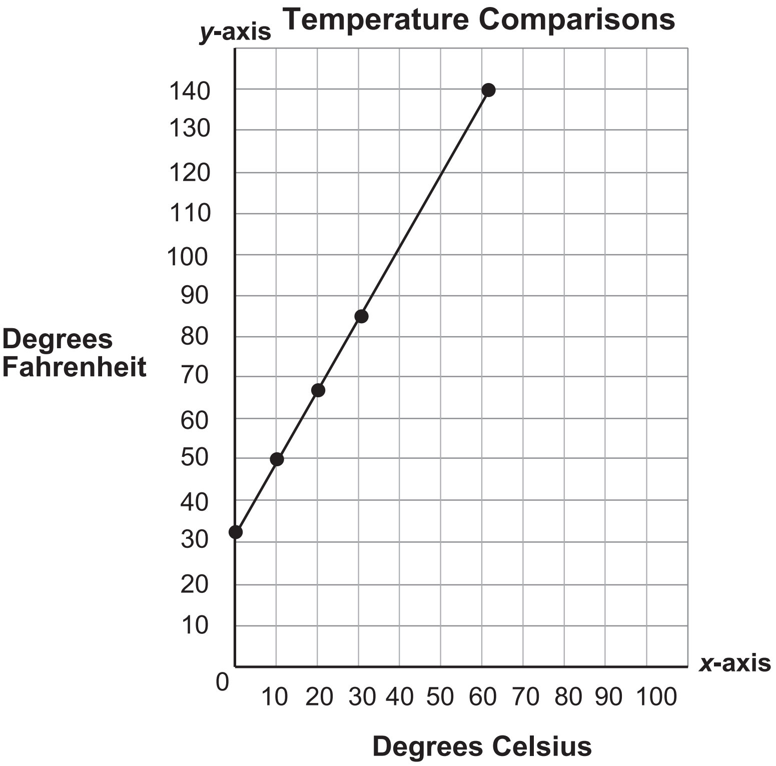
Interpreting Data and Lines of Best Fit

In many situations, the relationship between two variables may be of interest. The following table shows the relationship between Fahrenheit and Celsius temperatures. For instance, a temperature of 0 degrees Celsius corresponds to a temperature of 32 degrees Fahrenheit. This is when water freezes.

Temperature Comparisons

C	0	10	20	30
F	32	50	68	86

Let's graph these ordered pairs. We will graph the Celsius temperatures on the *horizontal* (↔) *x-axis* and the corresponding Fahrenheit temperatures on the *vertical* (↕) *y-axis*. We will carefully draw a line that contains these points.



Note that when the Celsius temperature is 60 degrees, the corresponding Fahrenheit temperature is close to 140 degrees, which happens to be the correct answer. When the temperature is 80 degrees Fahrenheit, what would the corresponding Celsius temperature be? We hope you guessed that the corresponding Celsius temperature is around 27 degrees.

The graph on the previous page is a visual representation of **data**. Such a graph is called a **line graph**. The *line graph* shows change over time. The *data* used is information in the form of numbers gathered for statistical purposes. The line graph, which is one type of **data display**, allows a person to see patterns that may not be obvious from just the equation. A graph is a powerful visual tool for representing data.

Practice

Answer the following.

1. To test a heart medicine, a doctor measures heart rate in beats per minute (bpm) of 5 patients before and after they take the medication. See the table below.

Patients' Heart Rate in Beats per Minute

Before	x	85	80	85	75	90
After	y	75	70	70	80	80

The data from this table is graphed on a **scatterplot (or scattergram)** on the following page. A *scatterplot* is a graph of data points used to observe a relationship between two numbers.

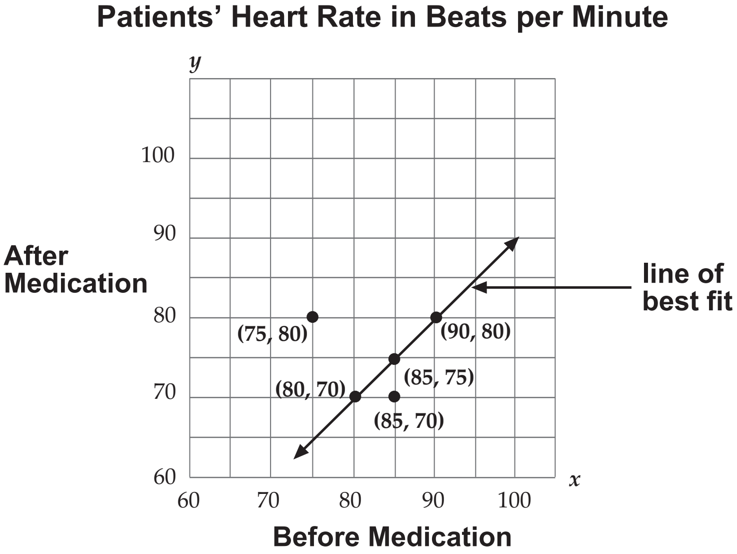
Notice that the points on the following scatterplot do not make a straight line. However, some of the points on a scatterplot can come close to forming a line. This line is called a **line of best fit**. A *line of best fit* may pass through some of the points, none of the points, or all of the points. A line of best fit may also be called a *trend line*. On the following page, the line drawn on the scatterplot contains the *majority* of the points.

The ordered pairs representing points on the line can be used to make predictions. The line of best fit shows if the correlation between the two variables is *weak* or *strong*. The correlation is strong if the data points come close to, or lie on, the line of best fit. If the data points do not come close to the line, the correlation is weak.

Use the line of best fit to estimate the following:

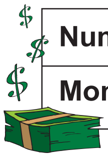
If the heart rate of a patient was 95 beats per minute *before* medication, what would it be after medication?

____



2. Six people were interviewed to see if there was a connection between the number of years they spent in college and their monthly salary. See the table below.

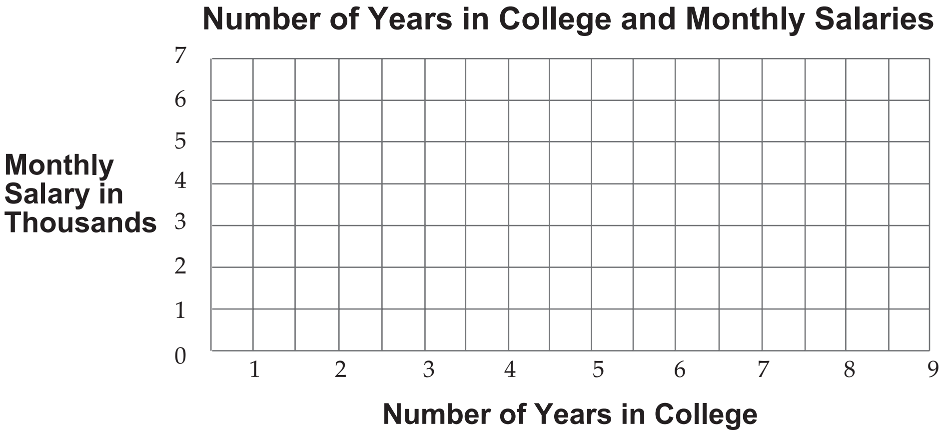
Number of Years in College and Monthly Salaries



Number of years in college	4	4	5	4	6	8
Monthly salary in thousands	3	3.5	4	2.5	5	5.5

a. Plot the data in ordered pairs on the graph below.

b. Draw a line of best fit.

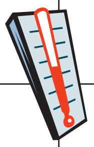


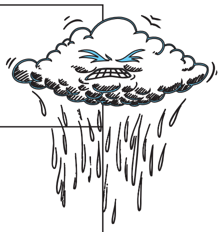
c. Predict the salary for a person with only 2 years of college.

____

3. The average February temperature and rainfalls for eight European cities are given in the table below.

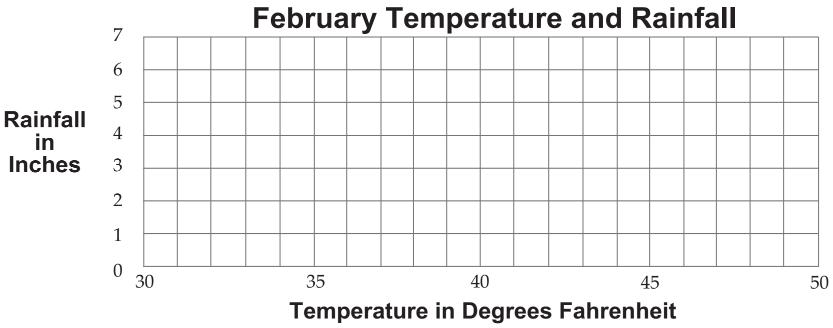
February Temperature and Rainfall





City	Temperature in degrees Fahrenheit	Rainfall in inches
Rome	47	3.3
Reykjavik	32	2.7
Paris	37	1.5
London	40	2.0
Dublin	41	2.7
Copenhagen	33	1.6
Berlin	31	1.9

a. Plot the data on the graph below.



b. Describe any pattern you see in the graph. ________________

____

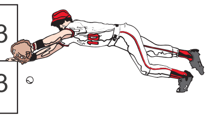
4. The table below gives the number of people in millions who attended regular season American League baseball games from 1987 through 1993.

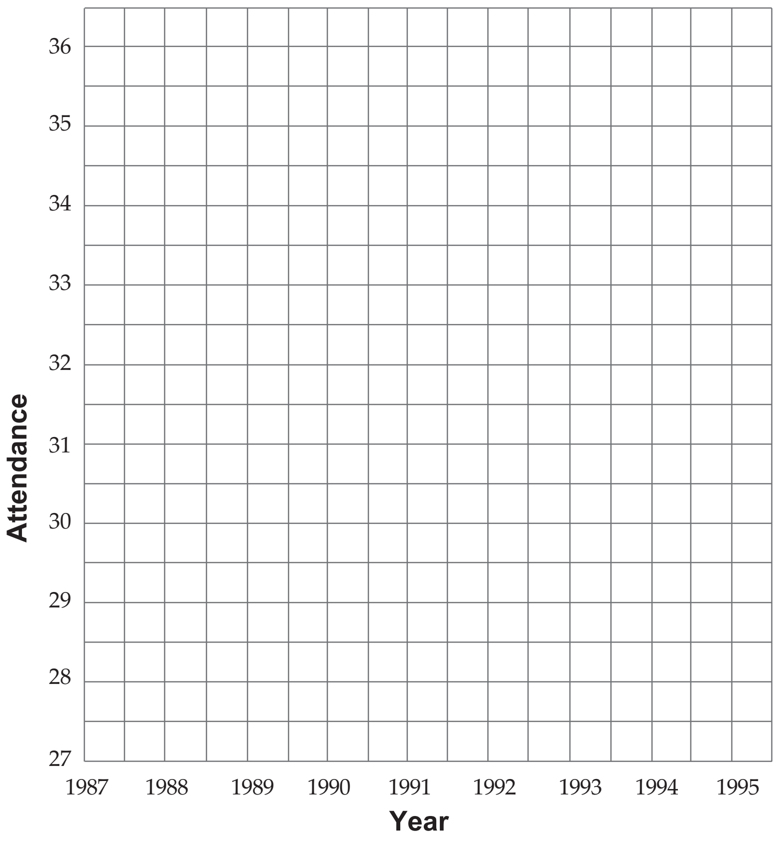
   a. Plot the data on the graph below.

   b. Draw a line of best fit.

American League Baseball Game Attendance 1987-1993

Year	1987	1988	1989	1990	1991	1992	1993
Attendance	27.3	28.5	29.8	30.3	32.1	31.8	33.3





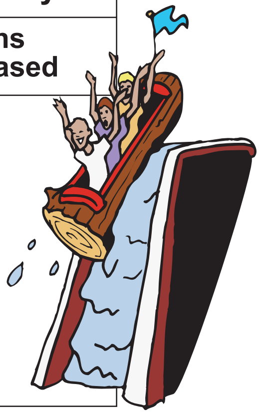
American League Baseball Game Attendance 1987-1993

   c. Estimate 1995 attendance. ______________________________

5. At a theme park in Florida, a log ride releases water in cycles as shown in the table below.

Water Release per Six-Minute Cycle

Cycle Number	Total Gallons of Water Released
1	65,000
2	130,000
3	195,000
4	260,000
5	?



If the pattern in the table continues, how many gallons of water will be released during cycle 5? Circle the letter of the correct answer.

a. 195,000 gallons
b. 325,000 gallons
c. 650,000 gallons
d. 1,300,000 gallons

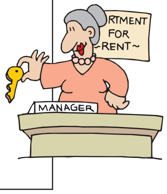
6. The Arvida Company has developed plans to build a large apartment complex in Svea, Florida. The developers found the following population listings for Svea over the last 30 years.

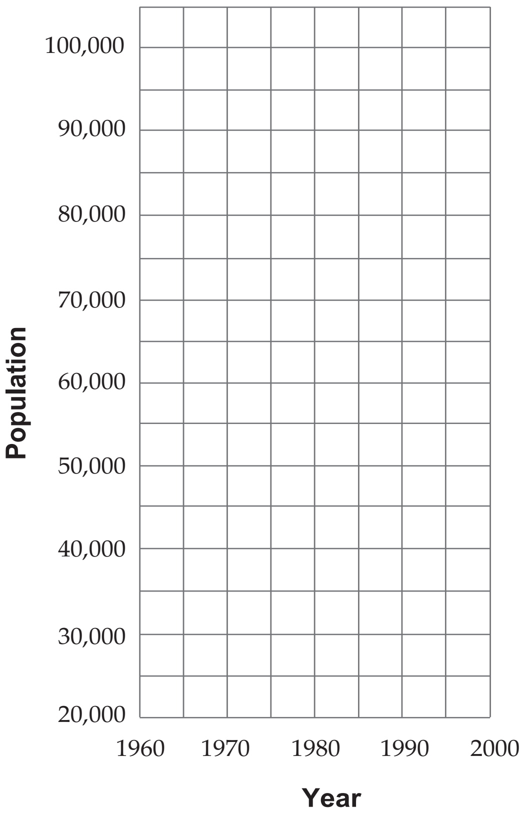
   a. Plot the data on the graph below.

   b. Draw a line of best fit.

Population of Svea, Florida 1960-1990

Year	Population
1960	25,000
1970	41,500
1980	65,600
1990	81,500




Population of Svea, Florida 1960-1990

c. What would be a good projection for the population in the year 2000? ____

d. What are some shortcomings in using this information to draw inferences? ____

7. Maria paints pictures. She sells small ones for \$20 and large ones for \$40. She wishes to make \$800 for her holiday gift fund. If x represents the number of small pictures that she sells and y represents the number of large pictures, then the equation which models this situation would be $20x + 40y = 800$.

   a. Complete the table below.

   b. Graph the situation.

   c. Draw a line of best fit.

Pictures to Sell for Holiday Gift Fund

Equation	x	**Substitute x**	**Solve for y**	**Solution**
$20x + 40y = 800$	0	$20(0) + 40y = 800$	20	(0, 20)
	20	$20(20) + 40y = 800$	________	________
	40	$20(40) + 40y = 800$	________	________

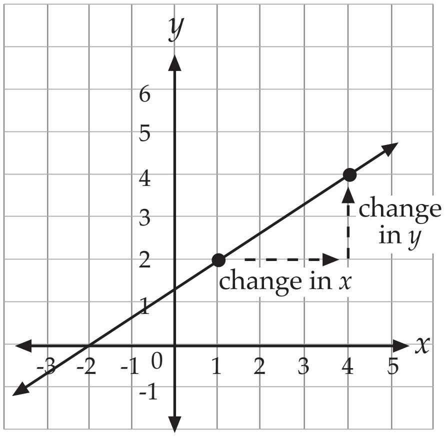
Pictures to Sell for Holiday Gift Fund

d. How many large pictures must she sell to reach her goal if she sells 30 small pictures? ________________________________

e. How many small pictures must she sell if she sells 20 large prints? ____

Slope

We all know that some mountains are steeper than others. Lines in a *coordinate plane* also have steepness. In math, the steepness of a line is called its **slope**. The *vertical* (↕) change is called the *change in y* and the *horizontal* (⟷) change is called the *change in x*.



Two Ways to Find the Slope of a Line

The slope or steepness of a line can be found using two points from the line. We will explore two definitions of slope and ways to find the slope of a line.

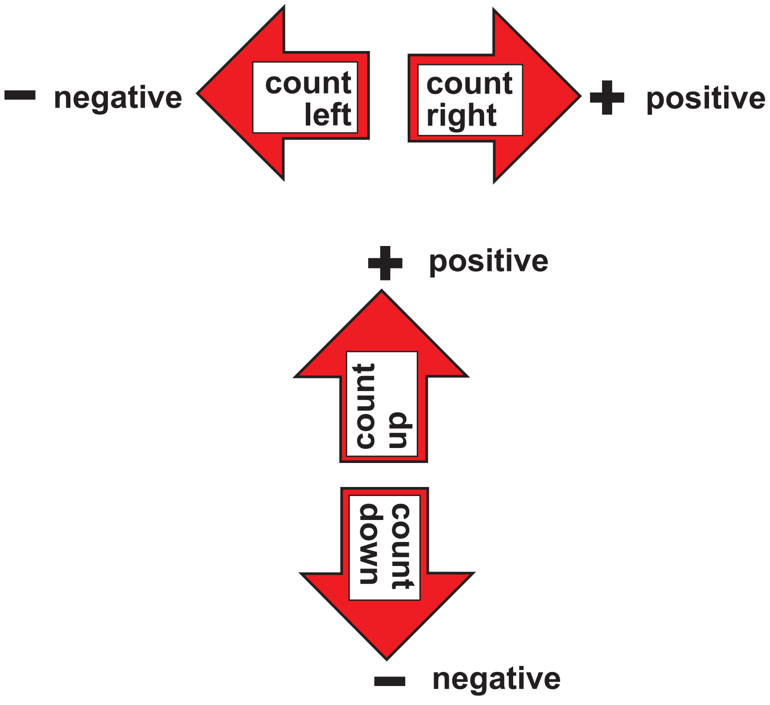
First, *slope* is defined to be the following **ratio**:

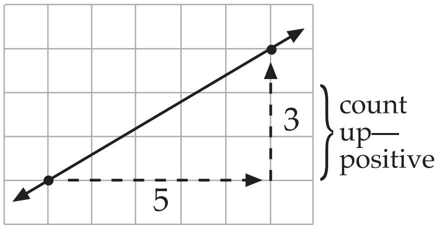
$$\text{slope} = \frac{\text{change in } y}{\text{change in } x}$$

A *ratio* is the quotient of two numbers used to compare quantities.

Example 1: Find the *slope* of the following six lines.

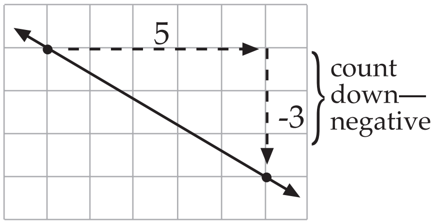
Solution: Find the *change in y* and the *change in x* by counting units. If you count to the right, that is a positive direction. If you count to the left, that is a negative direction. If you count up, that is a positive direction. If you count down, that is a negative direction. Pick any two points on the line, and travel from one point to the other.





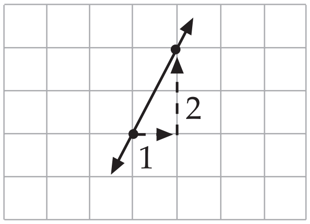
line 1

a. slope = $\frac{\text{change in } y}{\text{change in } x} = \frac{3}{5}$



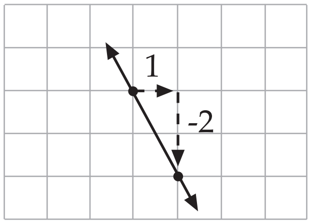
line 2

b. slope = $\frac{\text{change in } y}{\text{change in } x} = -\frac{3}{5}$



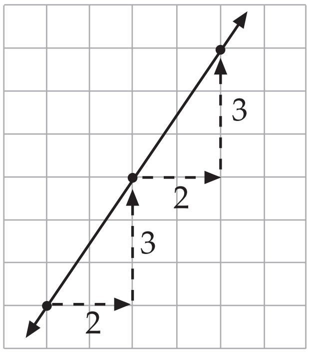
line 3

c. slope = $\frac{2}{1}$ or 2



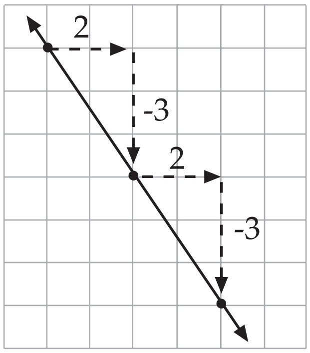
line 4

d. slope = $-\frac{2}{1}$ or -2



line 5

e. slope = $\frac{3}{2}$



line 6

f. slope = $-\frac{3}{2}$

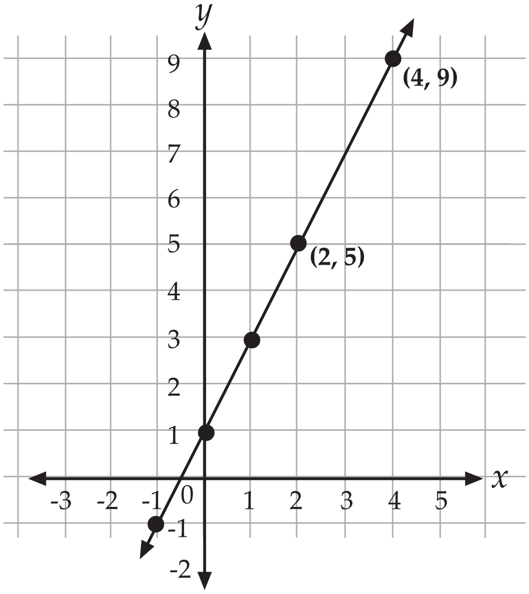
Another Way to Find the Slope of a Line

Here is another way to find the slope of a line. Pick any two points on a line. We will refer to the points as (x_1, y_1) and (x_2, y_2). The little numbers are called *subscripts* and they are just a way of letting the reader know that we are talking about *point 1* and *point 2*, and that they are *different* points. The subscripts mean that these are two distinct points of the form (x, y).
Note: The subscripts are *not* exponents.

Here is a *second* definition of slope:

$$\text{slope} = \frac{y_2 - y_1}{x_2 - x_1}$$

Example 2: Find the slope of this line.



Solution: Let (x_1, y_1) be (2, 5) and let (x_2, y_2) be (4, 9)

Substitute into the formula:

$$\text{slope} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{9 - 5}{4 - 2} = \frac{4}{2} = \frac{2}{1} \text{ } or \text{ } 2$$

Example 3: Find the slope of the line passing through (-7, 5) and (8, -2).

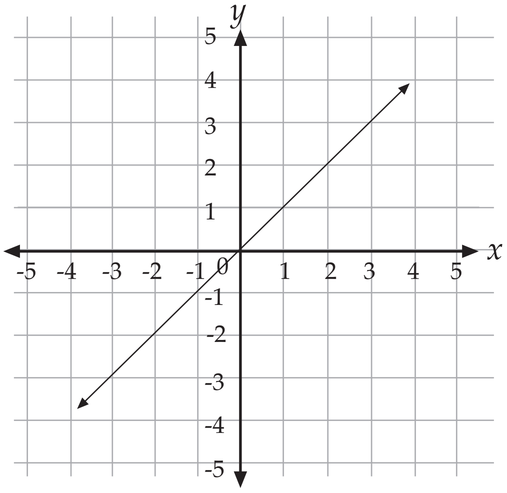
Solution: Let (x_1, y_1) be (-7, 5) and let (x_2, y_2) be (8, -2)

$$\text{slope} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-2 - 5}{8 - -7} = \frac{-7}{15}$$

Look to Be Sure—Positive and Negative Slopes

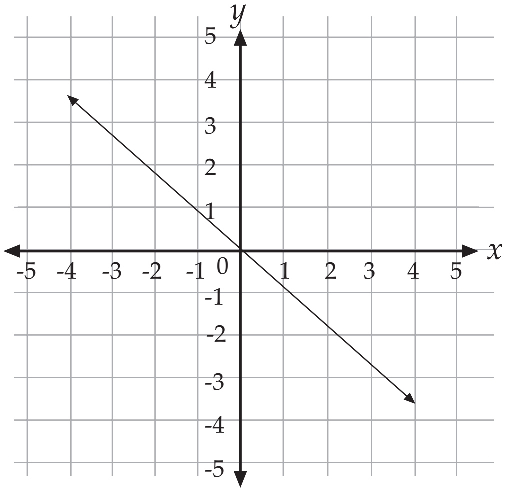
Look at the two graphs below. You can tell if the slope is positive or negative by looking. A line with a *positive slope* rises as the value of *x* increases. (See line 1 below.) A line with a *negative slope* falls as the value of *x* increases. (See line 2 below.)

line 1



positive slope

line 2



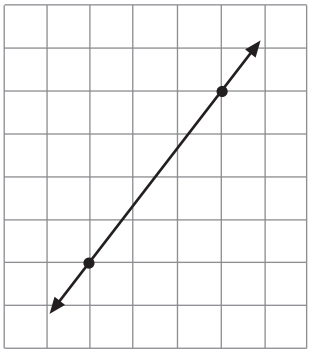
negative slope

Practice

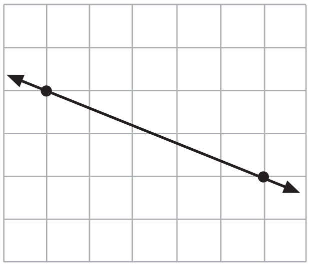
Find the **slope** *of the following lines using the first definition*:

$$\text{slope} = \frac{\text{change in } y}{\text{change in } x}$$

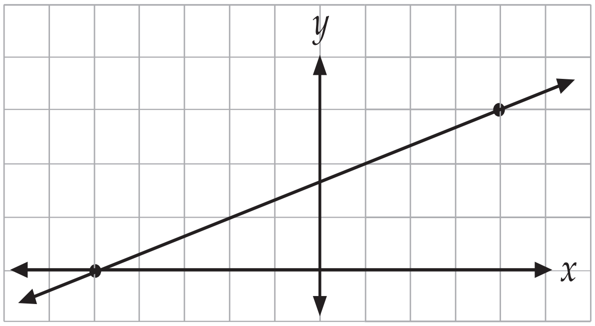
Write each answer in **simplest form**.



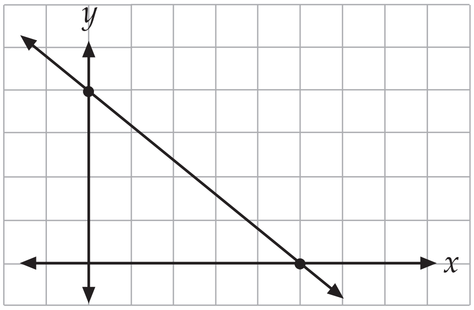
1. ____________



2. ____________



3. ____________



4. ____________________

Find the **slope** *of the following lines using the second definition*:

$$\text{slope} = \frac{y_2 - y_1}{x_2 - x_1}$$

Write each answer in **simplest form**.

5. passing through (0, 5) and (2, 6) ____________________

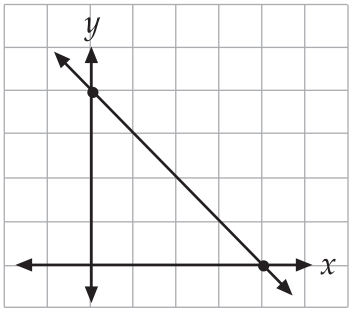
6. passing through (3, 4) and (4, 3) ____________________

7. passing through (1, -2) and (-1, -8) ____________________

8. passing through (0, -1) and (1, -7) ____________________

Circle the letter of the correct answer.

9. Find the slope of the line below.



   a. 4
   b. 2
   c. 1
   d. -1

10. Find the slope of the line through (-2, 8) and (2, -8).

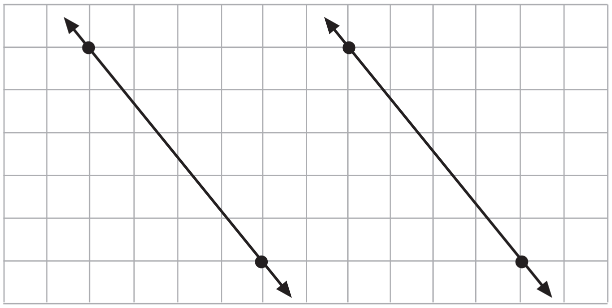
   a. -4
   b. $-\frac{1}{4}$
   c. 0
   d. 4

Answer the following.

11. These two lines are parallel. Find the slope of each line. __________

    What did you find out? ____

    ____

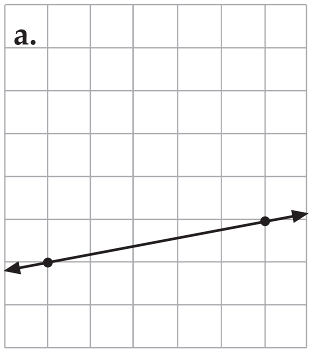


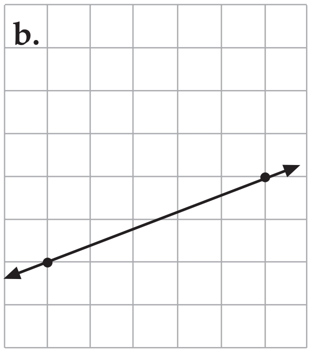
12. Find the slope of each line below.

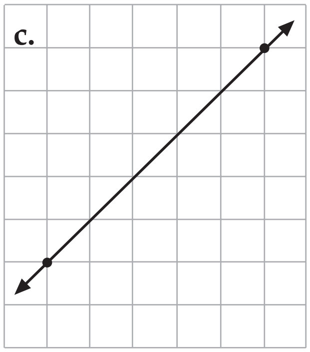
a. ______________

b. ______________

c. ______________







Notice that as you move from left to right the lines get steeper and steeper.

d. What do you notice about their slopes? ______________________

____

____

13. Would you rather climb a mountain with slope 8 or one with slope $\frac{1}{8}$? ________________________________

Explain why. ________________________________

14. Slope can also be thought of as a rate of change. In 1988 Florence Griffith Joyner set the women's world record for the 100-meter dash. Here are her statistics:

Joyner's 100-Meter Statistics in 1988

Seconds	Meters
2	19
4	38
6	57
10	95

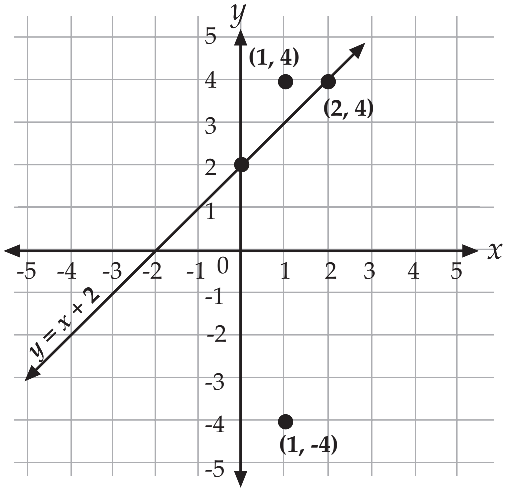
Find the slope of the line formed by the above points. Write the answer in *simplest form*.

____

Hint: Pick any two points to determine the slope. Then reduce the fraction so that the denominator is 1. This will tell you how many meters Ms. Joyner ran in 1 second.

Graphing Inequalities

Consider the line $y = x + 2$. We can think of the line separating the coordinate plane into three distinct parts.



- We have points above the line.
- We have points on the line.
- We have points below the line.

This awareness of the relative position of points with regard to the line is important when we need to solve **inequalities** like $y > x + 2$.

Remember: Inequalities state that one expression

—is greater than (>),
—is greater than or equal to (≥),
—is less than (<),
—is less than or equal to (≤), or
—is not equal to (≠)

another expression.

Let's investigate which points work in the above inequality. The *solution of an inequality with two variables* is an ordered pair of numbers that makes the inequality true. Are the points on the line solutions? For example, (2, 4) lies on the line. Does it make a true sentence when we substitute it into the inequality?

$y > x + 2$
$4 > 2 + 2$
$4 > 4$ This is false, so (2, 4) is *not* a solution!

If we investigate further, it turns out that *no* point *on* the line would work.

What about (1, -4)? This point was picked randomly, and it lies in the region below the line. Let's substitute into the inequality and see if this point is a solution.

$y > x + 2$
$-4 > 1 + 2$
$-4 > 3$ This also turns out to be *false*!

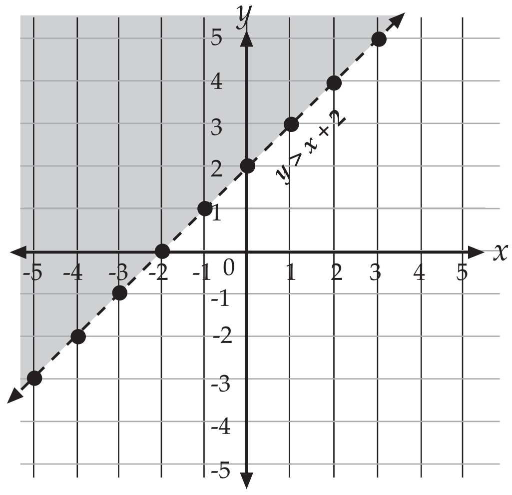
If we investigate further, it turns out that *no* point in the region *under* the line works.

Finally, we need to check and see if a point from the region above the line works. The point (1, 4) is such a point.

$y > x + 2$
$4 > 1 + 2$
$4 > 3$ This turns out to be true!

It also turns out that any point *above* the line works in the inequality.

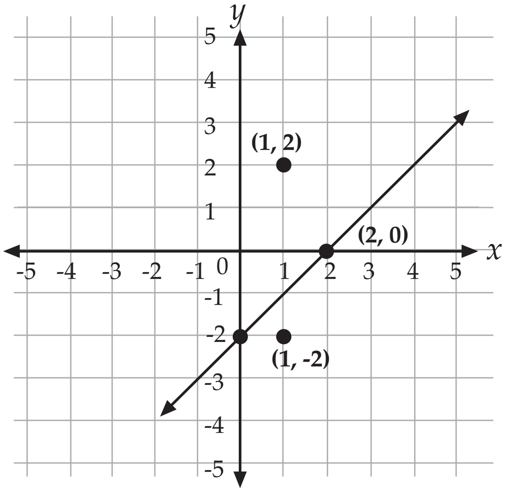
To show that these points are solutions, we shade the region above the line. Since no points on the line worked, we make the line *dotted*.



Example: Graph $y \le x - 2$

Solution:

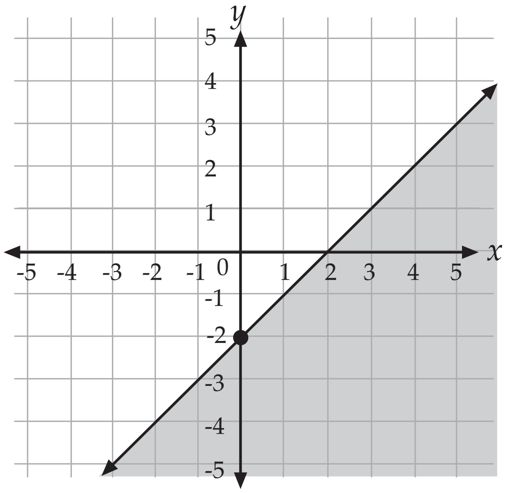
1. Graph the line $y = x - 2$. This will divide the coordinate plane into three regions—points *on*, *above*, and *below* the line.



2. Pick any point above the line, on the line, and below the line.
3. Substitute the points into the inequality, and see which ones work:

above the line	**on the line**	**below the line**
(1, 2)	(2, 0)	(1, -2)
$y \le x - 2$ $2 \le 1 - 2$ $2 \le -1$	$y \le x - 2$ $0 \le 2 - 2$ $0 \le 0$	$y \le x - 2$ $-2 \le 1 - 2$ $-2 \le -1$
This is false!	This is true!	This is true!

4. Make the line *solid*, since the test point (2, 0) was a solution. That means that *all* points on the line work. Shade below the line, since the test point (1, -2) was also a solution. That means that *all* points below the line are solutions.

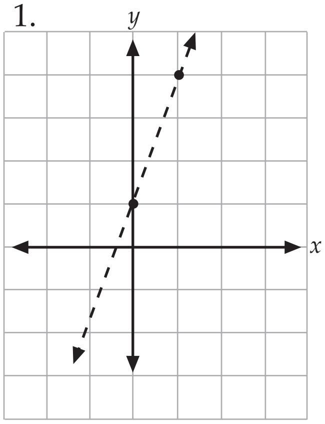


When solving inequalities, you will either shade above the line *or* below the line. Never both.

Practice

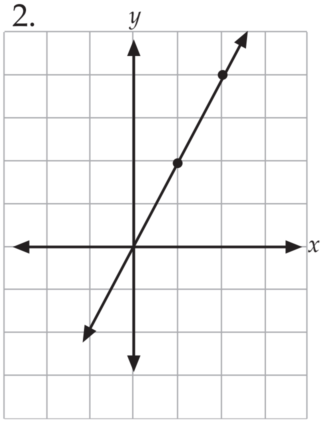
Pick any **point above and below** *the line.* **Substitute** *the points into the inequality. Shade the region that represents the* **solution of the inequality**.

1.



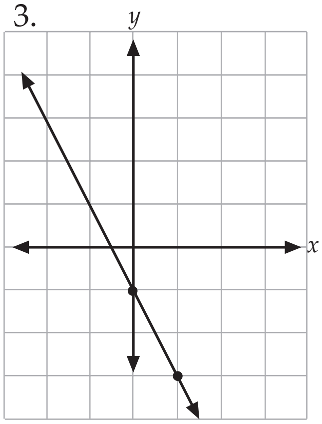
$y < 3x + 1$

2.



$y \geq 2x$

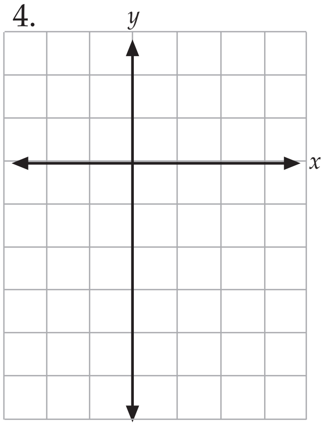
3.



$y \leq -2x - 1$

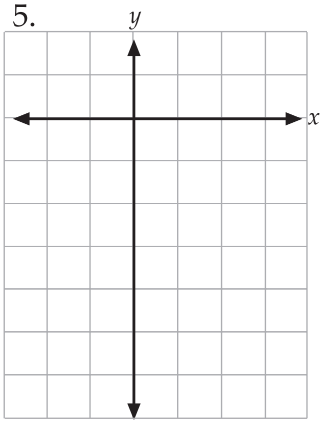
Graph each **inequality**. *Pick any points* **on, above, and below** *the line.* **Substitute** *the points into the inequality. Shade the region that represents the* **solution of the inequality**.

4.



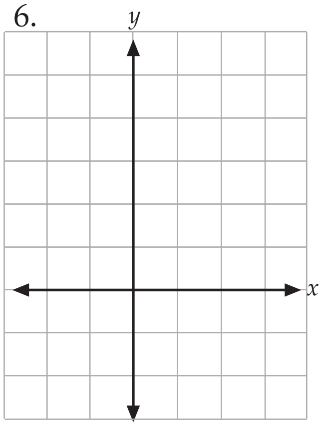
$y > -4x$

5.



$y \leq -3x - 3$

6.



$y \geq x$

Circle the letter of the correct answer.

7. A marathon is a long-distance race that covers 26.2 miles. You are running in a marathon. You realize that to finish the marathon, you must walk part of the way. Let x represent the number of miles that you must walk and y the number of miles you must run. Since you are not sure that you can finish the marathon, which of the following best describes your situation?

   a. $x + y > 26.2$
   b. $x + y \geq 26.2$
   c. $x + y < 26.2$
   d. $x + y \leq 26.2$

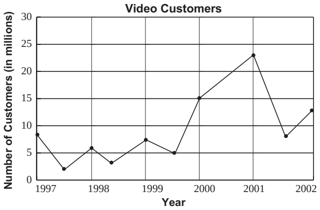
Practice

Match each definition with the correct term. Write the letter on the line provided.

______ 1. the quotient of two numbers used to compare two quantities

______ 2. information in the form of numbers gathered for statistical purposes

______ 3. a graph used to show change over time in which line segments are used to indicate amount and direction



______ 4. a graph of data points, usually from an experiment, that is used to observe the relationship between two variables

______ 5. line drawn as near as possible to the various points so as to best represent the trend being graphed; also called a *trend line*

______ 6. the steepness of a line, defined by the ratio of the change in y to the change in x

______ 7. a sentence that states one expression is greater than (>), greater than or equal to (≥), less than (<), less than or equal to (≤), or not equal to (≠) another expression

A. data

B. inequality

C. line graph

D. line of best fit

E. ratio

F. scatterplot (or scattergram)

G. slope

Lesson Two Purpose

- Understand and explain the effects of addition, subtraction, multiplication, and division on real numbers. (MA.A.3.4.1)
- Apply special number relationships such as sequences to real-world problems. (MA.A.5.4.1)

Sequences

In real life, we frequently see numbers that form a **pattern (relationship)**. That is, numbers which appear in a specific order (or pattern) determine a **sequence**.

Street signs in cities and towns use numbers such as 1st Avenue, 2nd Avenue, 3rd Avenue, etc. Likewise, house numbers form patterns with even numbers on one side of a street and odd numbers on the opposite side. If your house number is 1800, then your neighbor will most likely live at 1802 and the next neighbor at 1804, and so on.

When we shop for clothes and shoes, signs giving sizes are placed *in order* to help customers quickly locate needed sizes.

Example 1:

An apartment rents for $600 a month. Each year the monthly rent is expected to increase $25.00. What will the monthly rent be at the end of 4 years?

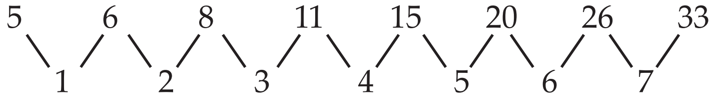
end of year	1	2	3	4
rent	$600	$625	$650	$675

+25 +25 +25

Notice that we could easily predict the monthly rent at the end of 5 years and at the end of 6 years.

Example 2:

Consider the numbers: 5, 6, 8, 11, 15, ?, ?, ? Can we predict the next 3 numbers? To see the pattern, we look at the difference between the numbers.



Example 3:

10^1	10^2	10^3	10^4	10^5
10	100	?	?	?

Using the pattern, what would be the value of 10^0? What would be the value of 10^6?

Answers:

$10^0 = 1$
$10^6 = 1{,}000{,}000$

Practice

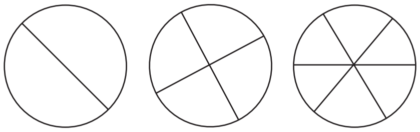
Write the **next three numbers** *of each sequence.*

1. 10, 12, 14, 16, ____________, ____________, ____________
2. 0.2, 0.4, 0.6, 0.8, ____________, ____________, ____________
3. 1, 4, 9, 16, ____________, ____________, ____________
4. 1, 2, 4, 8, ____________, ____________, ____________
5. 0, 4, 8, 12, ____________, ____________, ____________
6. 1, 8, 27, 64, ____________, ____________, ____________
7. 109, 104, 99, 94, ____________, ____________, ____________
8. 0, 3, 6, 9, ____________, ____________, ____________
9. 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, ____________, ____________, ____________
10. 3, 4.5, 6, 7.5, 9, ____________, ____________, ____________
11. Complete the following table.

3^1	3^2	3^3	3^4	3^5	3^6
3	9	____	____	____	____

12. Find the 12th term in the sequence 100, 98, 96, 94, … ____________
13. Find the 10th term in the sequence 1, 4, 9, 16, … ____________

14. Study the sequence of diagrams. Draw the 4th diagram in the sequence. Then complete the table.

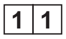


Lines	1	2	3	4	*n*
Regions	2	4	___	___	___

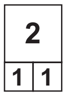
15. A special sequence is called the *Fibonacci sequence*. The sequence is named after Leonardo Fibonacci, an Italian mathematician and businessman who presented it in 1201.

We can make a picture showing the Fibonacci numbers 1, 1, 2, 3, 5, 8.

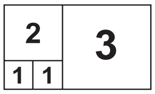
- Start with two small squares of size 1 next to each other.



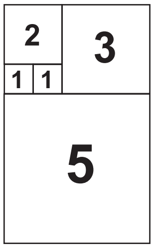
- On top of both of these, we draw a square of size 2 (= 1 + 1).

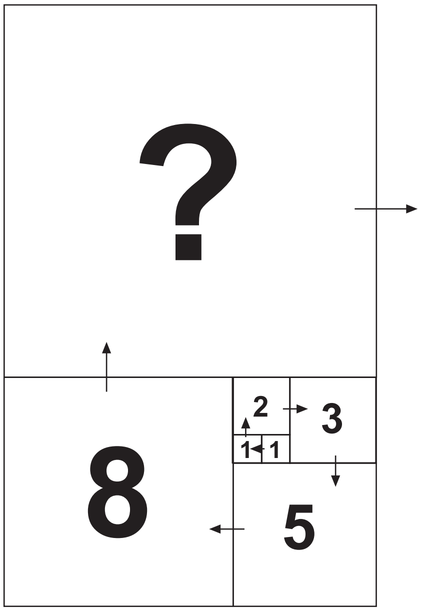


- We now draw a new square touching both the length of square 1 and square 2 (= 1 + 2).



- The next square drawn touching the length of square 2 and 3 (= 2 + 3).





We can continue adding squares around the picture, each new square having a **side** which is as long as the **sum** of the last two square lengths.

Fibonacci sequence arranged in a pattern of squares with sides whose lengths follow the sequence.

a. Find the next four numbers in the sequence 1, 1, 2, 3, 5, 8,

__________, __________, __________, __________ .

b. In your own words, describe the pattern of the sequence.

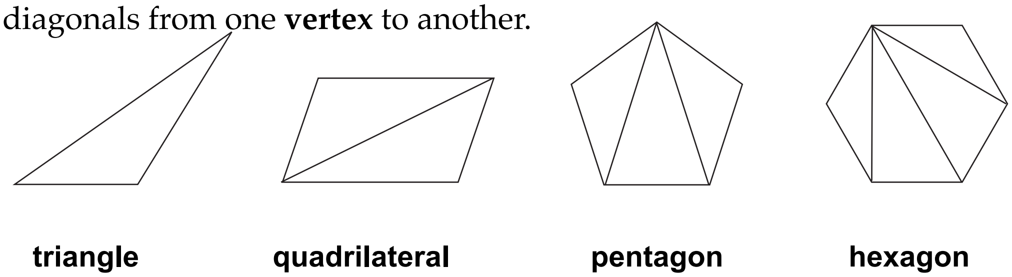
____

____

Investigating Patterns in Polygons

Now that we have studied geometric shapes and line graphs, we are going to investigate some interesting topics that combine shapes, graphs, and patterns.

Recall that the *sum* of the **measures (m) of an angle (∠)** of one **triangle** is 180 **degrees (°)**. Consider the **polygons** below and notice that we have divided the polygons (with 4 or more *sides*) into triangles by drawing diagonals from one **vertex** to another.



The **quadrilateral** can be divided into 2 triangles, which means that the sum of its **angles** is 180 x 2, or 360°. How many triangles are formed in the **pentagon**? How many triangles are formed in the **hexagon**? Let's use a chart to summarize our information.

Number of Sides, Triangles, and Degrees in Polygons

Polygon	Number of Sides	Number of Triangles	Sum of Angle Measures
Triangle	3	1	1 x 180 = 180°
Quadrilateral	4	2	2 x 180 = 360°
Pentagon	5	3	3 x 180 = 540°
Hexagon	6	4	4 x 180 = 720°

Do you see a pattern in the angle measures? Each sum is a result of multiplying 180 times the *number of sides minus 2*.

Algebraically, if n represents the number of sides of the polygon, we discover the formula: $180(n - 2)$ or $(n - 2)180$. The formula can be used to calculate the *sum of the measures of the angles* of any polygon.

Also, if the polygon's angles are all **congruent (≅)**, or the same shape and size, we can find the measure of each angle of the polygon. Recall from Unit 4 that if all angles in a polygon have the same measure, we describe the polygon as an **equiangular polygon**—a polygon that has equal angles.

If a quadrilateral is *equiangular*, we can find the *measure of each angle* by dividing the sum of the angles by the actual number of angles. A quadrilateral has 4 sides and 4 angles. To calculate the measure of each angle, we use the formula:

$$\frac{180(n-2)}{n}$$

Since a quadrilateral has 4 sides and 4 angles, then $n = 4$.

$$\begin{aligned}\text{each angle in a quadrilateral} &= \frac{180(4-2)}{4}\\ &= \frac{180(2)}{4}\\ &= \frac{360}{4}\\ &= 90^\circ\end{aligned}$$

Practice

Find the **sum of the measures of the angles** *of each polygon. Use the following formula:*

$$(n - 2)180.$$

Note: Names for polygons with more than 8 sides are not commonly used. For example, an 11-sided polygon may be referred to as an 11-gon.

1. pentagon ______________________________

2. hexagon ______________________________

3. **octagon** (8 sides) ______________________________

4. 20-gon ______________________________

5. 36-gon ______________________________

Find the **measure of each interior angle** *of the following polygons if each polygon is equiangular. Use the following formula:*

$$\frac{(n-2)180}{n}$$

6. triangle ____________________

7. pentagon ____________________

8. **dodecagon** (12 sides) ____________________

9. 36-gon ____________________

Complete the chart below to describe the **angle measures** *of the* **heptagon**, *the* **octagon**, *and the* **decagon**.

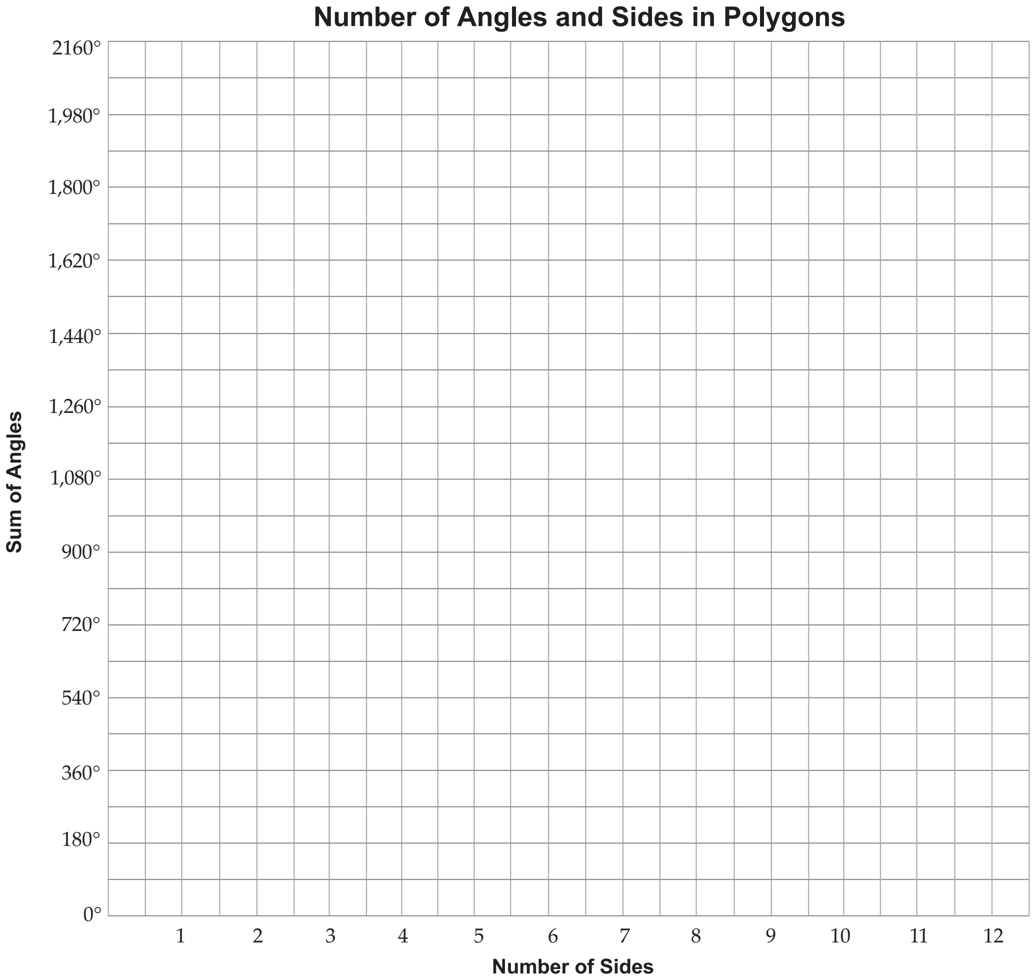
10.

Number of Sides, Triangles, and Degrees in Polygons

Polygon	Number of Sides	Number of Triangles	Sum of Angle Measures
Triangle	3	1	1 x 180 = 180°
Quadrilateral	4	2	2 x 180 = 360°
Pentagon	5	3	3 x 180 = 540°
Hexagon	6	4	4 x 180 = 720°
Heptagon	7		
Octagon	8		
Decagon	10		

11. a. Graph the results of problem 10.

Hint: Let the *x*-axis represent the number of sides. Let the *y*-axis represent the sum of the angles in increments of 180°.



b. Predict the sum of the angles of a **nonagon** (9 sides). ________

____

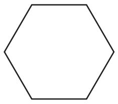
12. By using the line drawn for problem 11, predict the sum of the angles of a 12-gon. ______________________________________

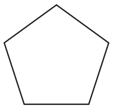
13. What happens to the *sum of the angles* as the number of sides increases? ______________________________________

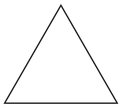
14. As the number of sides of a polygon increases, the shape becomes more and more like what geometric figure? ____________________

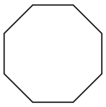
Practice

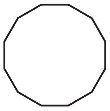
Match each **illustration** *with the correct term. Write the letter on the line provided.*

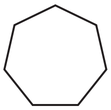
______ 1.  6 sides

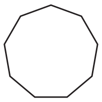
______ 2.  5 sides

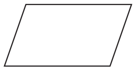
______ 3.  3 sides

______ 4.  8 sides

______ 5.  12 sides

______ 6.  7 sides

______ 7.  9 sides

______ 8.  4 sides

A. dodecagon

B. heptagon

C. hexagon

D. nonagon

E. octagon

F. pentagon

G. quadrilateral

H. triangle

Practice

Use the list below to write the correct term for each definition on the line provided.

angle (∠)
congruent (≅)
degree (°)
equiangular polygon
measure (m) of an angle (∠)
pattern (relationship)
polygon
sequence
side
sum
vertex

________________________ 1. a polygon with all angles equal

________________________ 2. the edge of a two-dimensional geometric figure

________________________ 3. figures or objects that are the same shape and the same size

________________________ 4. the common endpoint from which two rays begin or the point where two lines intersect

________________________ 5. an ordered list with either a constant difference (arithmetic) or a constant ratio (geometric)

________________________ 6. a predictable or prescribed sequence of numbers, objects, etc.

________________________ 7. the shape made by two rays extending from a common endpoint, the vertex

________________________ 8. the result of an addition

________________________ 9. the number of degrees (°) of an angle

________________________ 10. common unit used in measuring angles

________________________ 11. a closed plane figure whose sides are straight lines and do not cross

Lesson Three Purpose

- Determine probabilities using counting procedures, tables, and tree diagrams. (MA.E.2.4.1)
- Determine the probability for simple and compound events as well as independent and dependent events. (MA.E.2.4.2)

Tree Diagrams

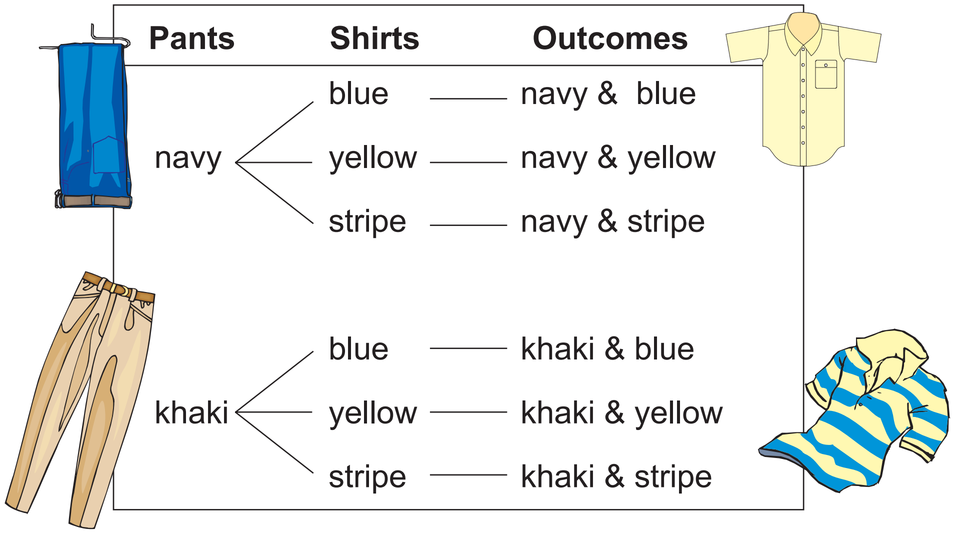
We can use a **tree diagram** to help us find the number of ways that one **event**, or possible *result* or **outcome**, can occur at the same time another *event* occurs. For example, we can consider different ways of combining garments to make outfits, or of looking at possible *outcomes* in games, or of ordering different combinations of items from a menu.

Example 1:

Suppose a school's colors are light blue and yellow. The school's uniforms allow khaki or navy pants to be worn with solid blue, solid yellow, or blue and yellow striped shirts. How many outfits are possible?

We can use a *tree diagram* to look at the various outfits. A tree diagram is a diagram that shows all possible outcomes of a given event.

School Uniform Color Outcomes

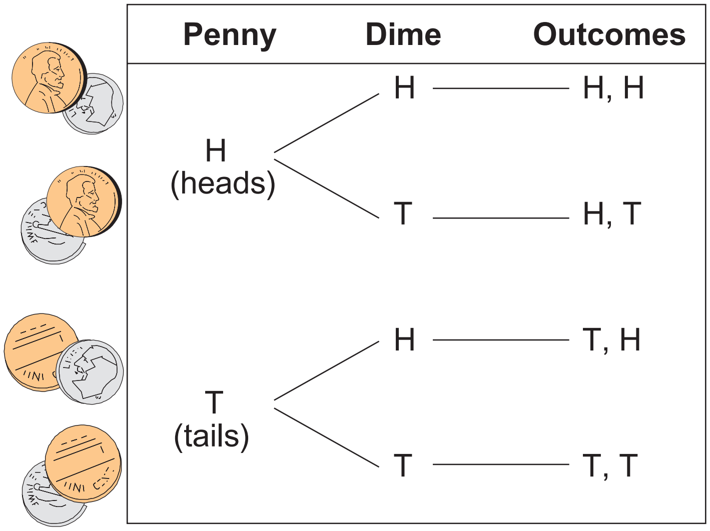


There are six possible outfits.

Example 2:

Suppose a penny and a dime are tossed at the same time. How many possible outcomes are there?

Coin Toss Outcomes



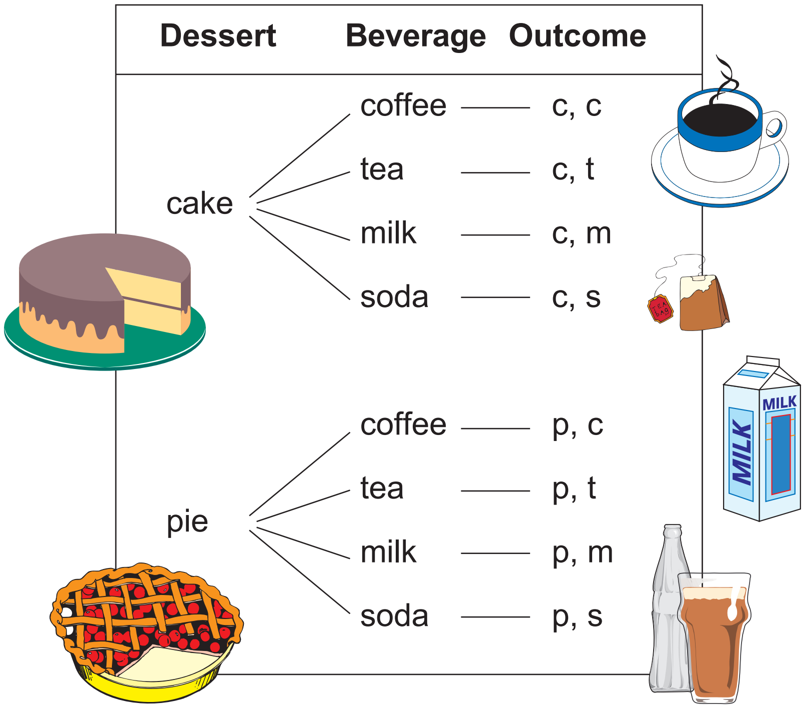
Notice that if *event A* has "a" outcomes and *event B* has "b" outcomes, then the total number of outcomes is (a)(b). The penny has 2 sides and the dime has 2 sides, so the outcomes are 2 x 2 = 4.

Look at the pants and shirts example with 2 choices for pants and 3 choices for shirts. The total number of outcomes was (2)(3) = 6.

Example 3:

Joe's Snack Shop serves 2 desserts: cake and pie. They also serve 4 beverages: coffee, tea, milk, and soda. If you choose 1 dessert and 1 beverage, how many possible combinations are there?

Joe's Snack Shop Outcomes



There are eight possible combinations.

Practice

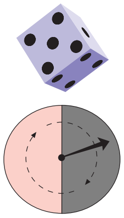
Draw a **tree diagram** *to solve each of the following.*

1. Jake has a choice of a solid, plaid, or striped shirt with a choice of tan or black pants. How many outfits can he wear?

   Answer: ______________________

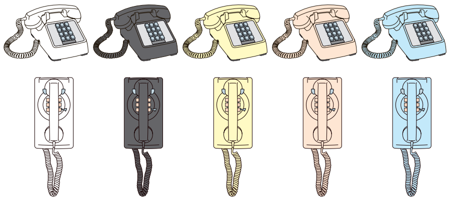
2. A six-sided die is rolled once and a spinner with one-half striped and one-half shaded is spun once. How many outcomes are there?

   Answer: ______________________



3. Telephones come in 2 styles, wall or desk. They come in 5 colors: white, black, yellow, tan, or blue. How many choices does a customer have?

   Answer: ______________________



Answer the following.

4. A snack bar serves 5 kinds of sandwiches and 3 kinds of juice. How many different orders for 1 sandwich and 1 juice could you place?

   Answer: ______________________

5. Janie has 5 blouses, 6 skirts, and 4 pairs of shoes. How many outfits are possible?

   Answer: ______________________

Circle the letter of the correct answer.

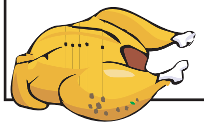
6. Jason is buying a car. He can choose from 3 different models, each offered in 4 different colors, and either a manual or automatic transmission. How many buying options are available?

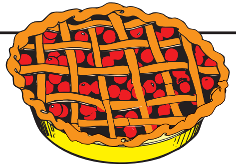
   a. 3
   b. 9
   c. 12
   d. 24

7. Suppose you are ordering a meal that consists of an entrée, a vegetable, a salad, and a dessert. Use the following menu to find how many different meals are possible.

Menu

Entrées	Vegetables	Salads	Desserts
roast beef	baked potato	green	cherry pie
chicken	wild rice	waldorf	cheesecake
grouper	green beans	carrot	vanilla ice cream
	broccoli		brownie
	cauliflower		apple cobbler
			key lime pie



a. 360
b. 18
c. 270
d. 180

Probability

At the beginning of a football game, a coin is tossed to determine which team will get the ball first. Let's say that Team A will get the ball first if the outcome is heads. Since the coin has two sides, it is **equally likely** that the coin will show heads or tails. *Equally likely* means that of the two possible outcomes—heads or tails—each has the same **probability** of occurring. *Probability* is the *ratio* of the number of favorable outcomes to the total number of outcomes.

$$\frac{\text{number of favorable outcomes}}{\text{total number of outcomes}}$$

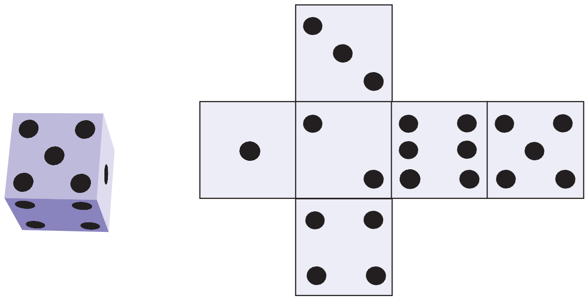
Let's look at the probability that Team A will get the ball first.

$$\text{probability of a a certain outcome} = \frac{\text{number of ways a certain outcome can occur}}{\text{number of possible outcomes}}$$

$$P(\text{heads}) = \frac{1}{2} \quad \frac{\text{one outcome of heads can occur}}{\text{there are 2 equally likely outcomes, heads or tails}}$$

Remember: *P* means probability; *P*(heads) refers to the probability of heads occurring.

Many games are played by rolling a die. Each die is in the shape of a **cube**. Recall from Unit 4 that a *cube* has six congruent sides and that each side is a square. Look at the die and the *flattened* **net** of the die below. The *net* is the plan which can be used to make a model of a solid. The net below forms a cube-shaped die.



If you rolled the die, there would be 6 equally likely outcomes. That means that each number has the same chance of landing face up.

Examples:

- Find the probability of rolling a 5.

$$P(5) = \frac{\text{number of ways of rolling a 5}}{\text{number of possible outcomes}}$$

$$P(5) = \frac{1}{6}$$

- Find the probability of rolling an odd number.

$$P(\text{odd}) = \frac{\text{number of odd numbers}}{\text{number of possible outcomes}}$$

$$P(\text{odd}) = \frac{3}{6} \quad \frac{\text{odd numbers are 1, 3, and 5}}{\text{there are 6 possibilities}}$$

$$P(\text{odd}) = \frac{1}{2} \quad \text{always simplify fractions}$$

- Find the probability of rolling either a 3 or a 4.

$$P(3 \text{ or } 4) = \frac{2}{6}$$

$$P(3 \text{ or } 4) = \frac{1}{3}$$

- Suppose you roll a die. What is the probability of rolling a 7?

$$\frac{\text{number of ways a 7 can occur:}}{\text{number of possible outcomes:}} = \frac{0}{6} = 0$$

$$P(7) = 0$$

- What is the probability of rolling a number from 1 through 6?

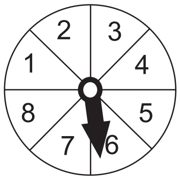
$$\frac{\text{number of ways a 1, 2, 3, 4, 5, or 6 can occur:}}{\text{number of possible outcomes:}} \frac{6}{6} = 1$$

$$P(1 \text{ through } 6) = 1$$

Note: An outcome that *will definitely happen* has a probability of 1. An outcome that *cannot happen* has a probability of 0.

Practice

The spinner is **equally likely** *to stop at any of the* **eight numbers***. The spinner is* **spun once***. Find the* **probability** *for each of the following. Write each answer in* **simplest form***.*



1. a seven ____________________
2. an even number ____________________
3. a two ____________________
4. an odd number ____________________
5. a 10 ____________________
6. a two, an eight, or a five ____________________
7. a number less than four ____________________
8. a number greater than six ____________________

Consider rolling one six-sided die. Find the **probability** *for the following. Write each answer in* **simplest form***.*

9. $P(4)$ ____________________
10. $P(1 \text{ or } 3)$ ____________________
11. $P(\text{even number})$ ____________________
12. $P(\text{prime number})$ ____________________
13. $P(\text{not } 6)$ ____________________

14. $P(10)$ ____________________

15. $P(4$ or less$)$ ____________________

16. $P($of a number less than $10)$ ____________________

In a bag, there are the following marbles:

- **3 blue marbles**
- **6 red marbles**
- **2 green marbles**
- **1 black marble.**

Suppose you select **one marble** *at random. Find the* **probability** *for each of the following.*

17. a green marble ____________________

18. a blue marble ____________________

19. a red marble ____________________

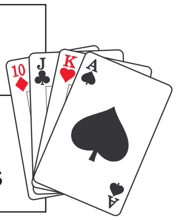
20. a black marble ____________________

21. a blue or red marble ____________________

22. a yellow marble ____________________

In a standard deck of **52 playing cards** *there are* **2 colors** *and* **4 suits**.

colors:	**red**	**black**
suits:	hearts diamonds	clubs spades



Each suit contains **all** *of the following* **13 cards**:

ace, **king**, **queen**, **jack**, **10**, **9**, **8**, **7**, **6**, **5**, **4**, **3**, *and* **2**.

Suppose you draw **one card** *from the deck. Find the* **probability** *of each of the following. Write each answer in* **simplest form**.

23. a jack ____

24. a red card _____________________________________

25. the queen of spades ______________________________

26. an eight, a nine, or a 10 ____________________________

27. a queen or a king ________________________________

28. a red or black card _______________________________

Answer the following.

29. In a group of 30 people, 26 are right-handed and the others are left-handed. If one person is selected randomly from the group, what is the probability that person will be left-handed?

    Answer: ____________

30. An ice chest contains 4 varieties of soda. There are 4 diet colas, 12 regular colas, 10 ginger ales, and 6 root beers. If the diet colas and the regular colas are the only types of soda with caffeine, what is the probability that a soda selected randomly from the ice chest contains caffeine?

Answer: ____________

Practice

Match each definition with the correct term. Write the letter on the line provided.

______ 1. a diagram in which all the possible outcomes of a given event are displayed

______ 2. two or more possible outcomes of a given situation that have the same probability

______ 3. the ratio of the number of favorable outcomes to the total number of outcomes

______ 4. a rectangular prism that has six square faces

______ 5. a possible result of a probability experiment

______ 6. a possible result or outcome in probability

A. cube

B. equally likely

C. event

D. outcome

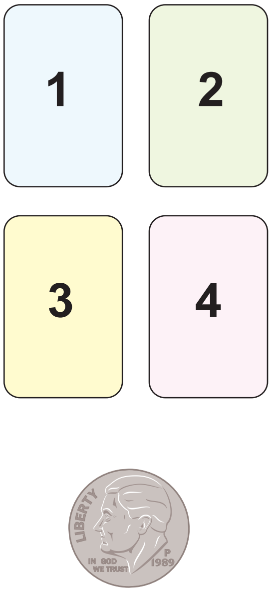
E. probability

F. tree diagram

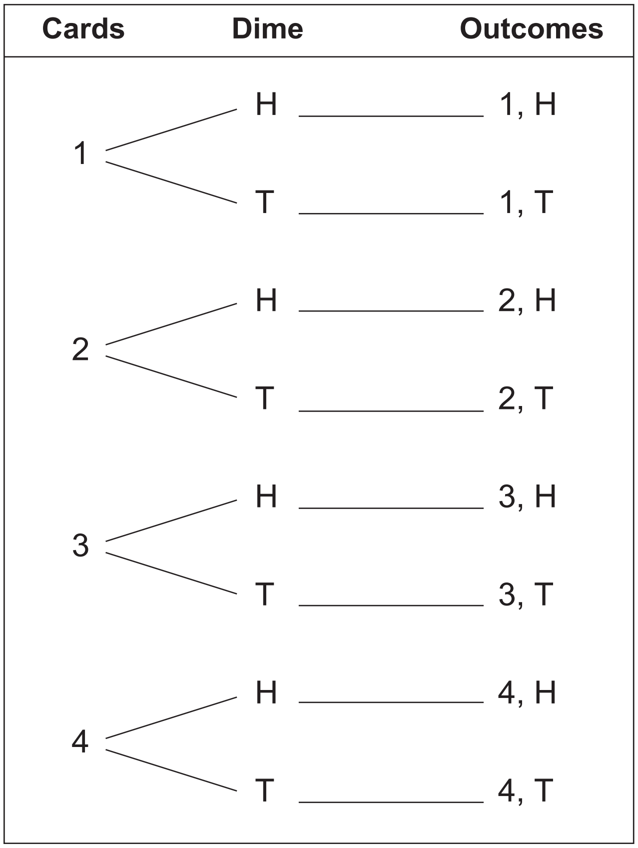
Probability of Independent Events

Think about first selecting a card and then tossing a coin. Since the outcome from selecting a card in no way affects the outcome of tossing the coin, these activities are called **independent events**. With *independent events*, the first event does *not* affect the outcome of the second event.

The tree diagram below shows all possible outcomes.



Independent Card and Coin Outcomes



There are eight possible outcomes.

Examples:

- The probability of drawing a 3 is 1 out of 4 chances or

  $P(3) = \frac{1}{4}$.

- The probability of tossing heads (H) is 1 out of 2 chances or

  $P(\text{H}) = \frac{1}{2}$.

- The possibility of drawing a 3 and then tossing heads is 1 out of 8 or

  $P(3 \text{ and H})$ is $\frac{1}{8}$ because $P(3) = \frac{1}{4}$ and $P(\text{H}) = \frac{1}{2}$ so

  $P(3 \text{ and H}) = P(3)$ times $P(\text{H})$ or $\frac{1}{4} \text{ x } \frac{1}{2} = \frac{1}{8}$.

Below is the formula for finding the probability of independent events.

Probability of Independent Events

The probability of two *independent* events (*A* and *B*) both occurring can be found by multiplying the probability of the first event (*A*) by the probability of the second event (*B*).

$$P(A \text{ and } B) = P(A) \text{ x } P(B)$$

Practice

Refer to the example on page 645. A **card** *is drawn from the cards shown below. Then a* **coin** *is tossed. Find the* **probability** *for each of the following. Write each answer in* **simplest form***. Use the formula:*

$$P(A \text{ and } B) = P(A) \text{ x } P(B)$$

1. *P*(2, H) ______________________________

2. *P*(3, T) ______________________________

3. *P*(even number, H) ______________________________

4. *P*(odd number, T) ______________________________

5. *P*(number less than 4, H) ______________________________

6. *P*(number greater than 1, T) ______________________________

7. *P*(**prime number**, H) ______________________________

Remember: A *prime number* (2, 3, 5, 7, 11, etc.) is any **whole number** with only two **factors**, 1 and itself. A *factor* is a number that divides evenly into another number. For example, factors of 20 are 1, 2, 4, 5, 10, and 20. However, by definition, the number 1 is excluded from the set of prime numbers.

A **coin** *is tossed and then a* **die** *is rolled. Find the* **probability** *for each of the following. Write each answer in* **simplest form***.*

8. *P*(H, 3) ______________________________

9. *P*(T, 5) ______________________________

10. *P*(H, 6) ______________________________

11. P(T, 2 or 3) ____________________

12. P(H, a number less than 5) ____________________

13. P(T, an odd number) ____________________

14. P(T, a prime number) ____________________

One bag *of marbles contains the following*:

- **3 red marbles**
- **4 white marbles**.

A **second bag** *of marbles contains the following*:

- **6 yellow marbles**
- **3 green marbles**.

One marble *is drawn from* **each bag**. *Find the* **probability** *of each of the following. Write each answer in* **simplest form**.

15. P(red, yellow) ____________________

16. P(red, green) ____________________

17. P(white, green) ____________________

18. P(white, yellow) ____________________

Draw a **tree diagram** *to find the* **probability** *for each of the following.*

19. If you toss a coin 3 times, what is the probability that you will get 3 heads? ____________

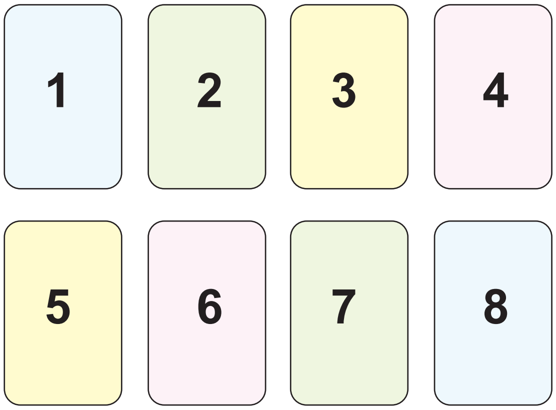
20. If you toss a coin 3 times, what is the probability that you will get either all heads or all tails? ____________

Challenge problem:

21. If a couple plans to have 4 children, what is the probability that all 4 will be girls? ____________

Probability of Dependent Events

Suppose these cards are shuffled and placed face down. What is the probability of drawing a 5?



We know that $P(5)$ is $\frac{1}{8}$ because the 5 is 1 card out of 8 cards. Suppose you draw the 5 and do *not* replace the card, and then draw a second card. What is the probability that the second card drawn will be a 2?

Since the first card was not replaced, the probability of drawing a 2 is $\frac{1}{7}$ because the 2 is 1 card out of the 7 cards left. The outcome of drawing the first card *affects* the outcome of drawing the second card from the remaining cards. Therefore, we describe the events of drawing a 5 and then drawing a 2 as being **dependent events**. With *dependent events*, the first event *affects* the outcome of the second event.

$$\begin{aligned} P(5, 2) &= \tfrac{1}{8} \text{ x } \tfrac{1}{7} \\ &= \tfrac{1}{56} \end{aligned}$$

So the probability of drawing a 5 and then drawing a 2 is $\frac{1}{56}$, or one chance out of 56.

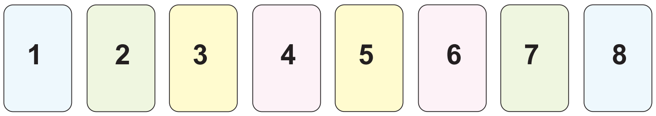
The Probability of Dependent Events

The probability of two *dependent* events (*A* and *B*) both occurring can be found by multiplying the probability of the first event (*A*) and the probability of the second event (*B*) after the first event (*A*) occurs.

$P(A$ and $B) = P(A) \times P(B$ following $A)$

Practice

Consider drawing **one card** *from the* **eight cards** *shown below.*



Then **without replacement**, *drawing a* **second card**. *Find the* **probability** *for each of the following. The cards will be face down and will have been shuffled. Write each answer in* **simplest form**. *Use the formula:*

$P(A)$ x $P(B$ following $A)$

1. $P(6, 1)$ ____

   Hint: $P(6)$ is $\frac{1}{8}$ and $P(1)$ is $\frac{1}{7}$

   so: $P(6, 1) = \frac{1}{8} \text{ x } \frac{1}{7} = \frac{1}{56}$

2. $P(6, \text{odd})$ ____

   Remember: An **odd number** (1, 3, 5, 7, 9, 11, …) is any whole number {0, 1, 2, 3, 4, …} *not* divisible by 2.

   Hint: $P(6) = \frac{1}{8}$ and $P(\text{odd}) = \frac{4}{7}$

   so: $P(6, \text{odd}) = \frac{1}{8} \text{ x } \frac{4}{7}$

3. $P(\text{odd}, 6)$ ____

4. P(3, prime) ____

   Hint: This one is tricky. Since the number 3 is a prime number and calculated in the first draw, do *not* include the number 3 in the number of prime numbers for the second draw.

5. P(even, 5) ____

   Remember: An **even number** (2, 4, 6, 8, 10, 12, …) is any whole number divisible by 2, with no leftovers.

6. P(odd, even) ____

In a bag there are the following marbles:

- **3 red marbles**
- **2 white marbles**
- **4 blue marbles**.

Once a marble is selected, it is **not replaced**. *Find the* **probability** *of each of the following. Write each answer in* **simplest form**.

7. P(red, white) ____

8. P(white, blue) ______________________________

9. P(white, white) ______________________________
Hint: Since the probability for one white marble is calculated in the first selection, do *not* include that same white marble in the second draw.

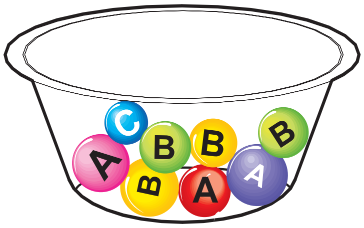
10. P(blue, blue) ______________________________

11. P(red, red) ______________________________

12. P(red, blue) ______________________________

Practice

Suppose you draw marbles from the bowl below **without replacement**. *Find the* **probability** *for each of the following. Write each answer in* **simplest form**.



1. $P(\text{A, B})$ ______________________

2. $P(\text{A, C})$ ______________________

3. $P(\text{B, A, C})$ ______________________
   Hint: $P(\text{B}) = \frac{4}{8}$, $P(\text{A}) = \frac{3}{7}$, $P(\text{C}) = \frac{1}{6}$

4. $P(\text{B, A, A})$ ______________________
   Hint: $P(\text{B}) = \frac{4}{8}$, $P(\text{A}) = \frac{3}{7}$, $P(\text{A}) = \frac{2}{6}$

5. $P(\text{B, B, A})$ ______________________

6. $P(\text{A, A, A})$ ______________________

Practice

Use the list below to write the correct term for each definition on the line provided.

dependent events	**odd number**
even number	**prime number**
factor	**simplest form**
independent events	**whole number**

________________________ 1. two events in which the outcome of the first event does *not* affect the outcome of the second event

________________________ 2. two events in which the first affects the outcome of the second event

________________________ 3. any whole number with only two factors, 1 and itself

________________________ 4. any whole number divisible by 2

________________________ 5. a number or expression that divides exactly another number

________________________ 6. any number in the set {0, 1, 2, 3, 4, …}

________________________ 7. any whole number *not* divisible by 2

________________________ 8. a fraction whose numerator and denominator have no common factor greater than 1

Lesson Four Purpose

- Understand and use the real number system. (MA.A.2.4.2)
- Interpret data that has been collected, organized, and displayed in charts, tables, and plots. (MA.E.1.4.1)
- Calculate measures of central tendency (mean, median, and mode) and dispersion (range) for complex sets of data and determine the most meaningful measure to describe the data. (MA.E.1.4.2)
- Explain the limitations of using statistical techniques and data in making inferences and valid arguments. (MA.E.3.4.2)

Measures of Central Tendency—Mean, Median, and Mode

The **mean**, **median**, and **mode** are sometimes called **measures of central tendency**. *Measures of central tendency* describe how *data* is *centered*. Each of these measures describes a set of data in a slightly different way.

- The **mean (or average)** is the sum of the data divided by the number of items.

  When the data is centered around the *mean*, this data is considered an appropriate measure of central tendency. The mean can be distorted by an *extreme* value, a value that is much greater than or less than the other values.

- The **median** is the middle item when the data is listed in numerical order. If there is an even number of items, the median is the average of the two middle numbers.

  When there is an extreme value in a set of data that distorts the mean, the *median* is an appropriate measure of central tendency.

- The **mode** is the item that appears most often. There can be more than one mode. There is *no* mode if each item appears only once.

  When data cannot be averaged (to find a mean) or listed in numerical order (to find a median), the *mode* is the appropriate measure of central tendency.

The **range** (of a set of numbers) is the difference between the highest and lowest value in a set of data. The *range* can help you decide if differences among the data are important.

Example 1:

Consider Larry's math test scores for the last 9 weeks:

0, 55, 70, 80, 80, 82, 85, 88, 100

Notice that the grades are in order from lowest to highest. Since there are three measures of central tendency and a range to use, we will consider each one to arrive at a fair grade for Larry.

- The *mean* (or average) is the sum of the numbers in the **set** divided by how many numbers are in the *set*. The mean is what most people think of when they think of finding an *average*. The set is the collection of numbers.

$$\frac{0 + 55 + 70 + 80 + 80 + 82 + 85 + 88 + 100}{9} = \frac{640}{9} = 71 \text{ rounded to nearest whole number}$$

Larry's mean or average grade would be 71.

- The *mode* is the *number that occurs most often* in a set of data. Sometimes there is no mode, and sometimes there are several modes.

  0, 55, 70, **80**, **80**, 82, 85, 88, 100

Larry's mode would be 80 because it appears twice.

- The *median* is the *number in the middle*. To find the median, the numbers have to be placed in numerical order.

  0, 55, 70, 80, **80**, 82, 85, 88, 100

In this case 80 is the median because it is the number in the middle.

Besides these three measures of central tendency, we can also consider the range of Larry's grades.

The *range* can be reported in one of two ways.

- The range can be reported as the *lowest value to the highest*.

  The range of Larry's grades is 0 to 100 because 0 is the lowest value and 100 is the highest value.

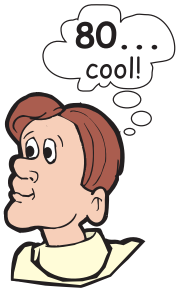
- The range can also be reported as the **difference** between the *highest and lowest values*.

  The range of Larry's grades is 100 because 100 is the difference between the smallest number (0) subtracted from the largest number (100) in the set.

Let's review what happened when we used the three measures of central tendency and the range:

mean	71
mode	80
median	80
range	100

Probably the fairest grade to give Larry for the 9 week grading period would be an 80. Most of the time Larry appears to be a "B" student, yet his average (mean) was a 71. The bottom two grades of 0 (for the test he missed) and 55 distorted the average. This is something that you need to be aware of when you read about averages in the paper.



Example 2:

Let's look at a second example. Below are the weights of 8 tenth grade girls:

90, 100, 110, 120, 124, 130, 132, 150

We will find the three measures of central tendency.

- *mean* (average)

  $$\frac{90 + 100 + 110 + 120 + 124 + 130 + 132 + 150}{8} = \frac{956}{8} = 120 \text{ rounded to nearest whole number}$$

- *mode* (the number which appears the most)

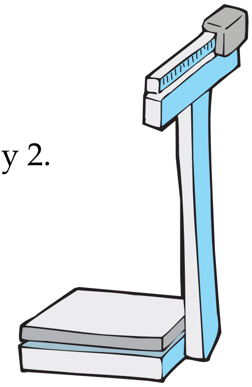
  Notice that all the numbers are different. Therefore there is no mode.

- *median* (the middle)

  90, 100, 110, 120, ↓ 124, 130, 132, 150

  To find the median, add 120 and 124, then divide by 2.

  $$\frac{120 + 124}{2} = \frac{244}{2} = 122$$



Summary: mean 120
mode .... not one
median 122

Both the mean and the median appear to be good representations of the data.

Practice

Answer the following.

1. These are the scores that Mrs. Landry's 4th period students earned on the last pre-algebra test:

   95, 90, 95, 75, 100, 80, 75, 80, 95, 100,

   70, 95, 70, 90, 95, 85, 90, 85, 90, 90,

   90, 100, 75, 100, 90, 75, 95, 100, 85, and 95.

   a. Find the mean. Round to the nearest *whole number*.

   ____

   Put the numbers in order.

   b. Find the mode. ______________________________________

   c. Find the median. ____________________________________

   d. What is the best number to describe the results of the test?

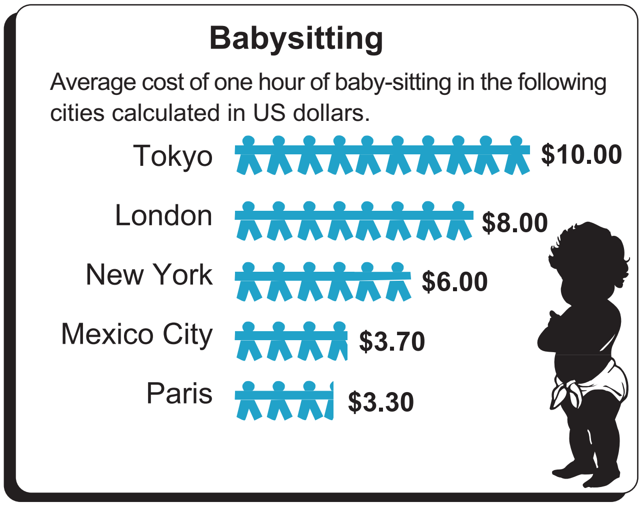
   ____

   ____

   e. The difference between the highest and lowest number is called

   the range of the data. Find the range. ___________________

2. Find the mean, mode, median, and range of the data below.



Source: Runzheimer International

mean ______________________________

mode ______________________________

median ______________________________

range ______________________________

3. Explain the following statement in your own words:
"The median price of a house in Florida is $130,000."

Explanation: ______________________________

____

____

____

____

____

4. At Apex Industries, 6 employees earned $22,000, 2 earned $30,000, and 3 earned $40,000. Find the mean, mode, and median.

mean (**round to the nearest dollar**) ______________________

mode ____

median ____

What would be the best description of the average salary at Apex

and why? ____

____

____

____

5. If your average monthly salary for 12 months is $3,400, how much

do you make per year based on this average? _____________

6. Jane is taking pre-algebra. Her teacher gives 5 tests per grading period. Jane has made an 80, 85, 60, and 70 on 4 of the tests. Her final average was exactly 74 (*without* rounding to the nearest whole number).

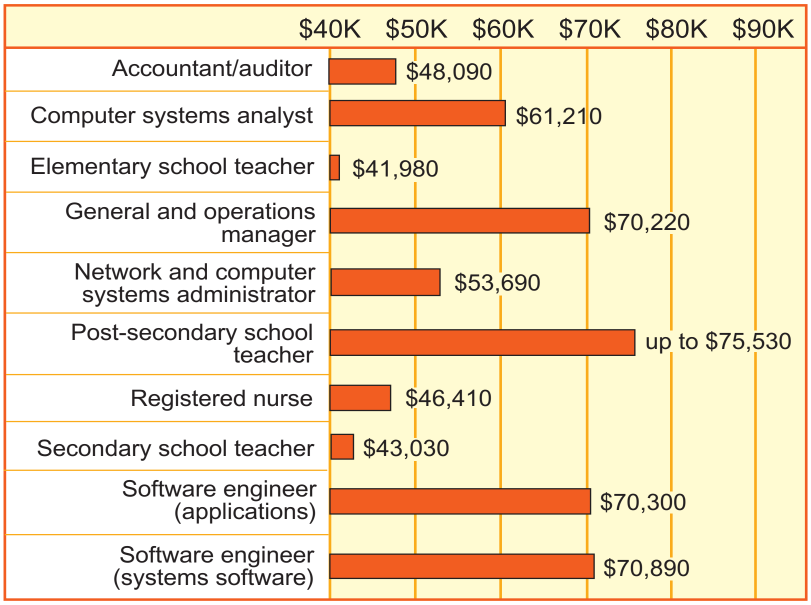
What did she make on her 5th test?

____

Hint: Think algebraically. Set up an equation to find the missing number.

7. Here are the 10 best-paying careers showing the most growth through 2010 with their average salaries as of 2000.

Ten Best-Paying Careers through 2010



Source: U.S. Department of Labor, Bureau of Labor Statistics

a. Do you think that these salaries are really accurate?

____

____

b. What factors are being omitted? ____________________

____

c. Find the mean salary in the table above. ______________

Practice

Use the list below to write the correct term for each definition on the line provided.

difference
even number
mean
measures of central tendency
median
mode
probability
range
set
tree diagram

________________________ 1. the middle point of a set of ordered numbers where half of the numbers are above the median and half are below it

________________________ 2. the result of a subtraction

________________________ 3. a collection of distinct objects or numbers

________________________ 4. the difference between the highest (H) and the lowest value (L) in a set of data

________________________ 5. the arithmetic average of a set of numbers

________________________ 6. the score or data point found most often in a set of numbers

________________________ 7. the mean, median, and mode of a set of data

________________________ 8. a diagram in which all the possible outcomes of a given event are displayed

________________________ 9. the ratio of the number of favorable outcomes to the total number of outcomes

________________________ 10. any whole number divisible by 2

Practice

Match each definition with the correct term. Write the letter on the line provided.

______ 1. line drawn as near as possible to the various points so as to best represent the trend being graphed; also called a *trend line*

______ 2. an ordered list with either a constant difference (arithmetic) or a constant ratio (geometric)

______ 3. a mathematical sentence that equates one expression to another expression

______ 4. the steepness of a line, defined by the ratio of the change in y to the change in x

______ 5. a sentence that states one expression is greater than, greater than or equal to, less than, less than or equal to, or not equal to another expression

______ 6. a polygon with five sides

______ 7. a polygon with six sides

______ 8. a straight line that is endless in length

______ 9. all points whose coordinates are solutions of an equation

______ 10. any value for a variable that makes an equation or inequality a true statement

______ 11. the location of a single point on a rectangular coordinate system where the digits represent the position relative to the x-axis and y-axis

A. equation

B. graph of an equation

C. hexagon

D. inequality

E. line

F. line of best fit

G. ordered pair

H. pentagon

I. sequence

J. slope

K. solution

Unit Review

Answer the following.

Write **yes** if the given ordered pair is a solution of $4x + y = 12$. Write **no** if the given ordered pair is *not* a solution.

__________ 1. (0, 12)

__________ 2. (2, 4)

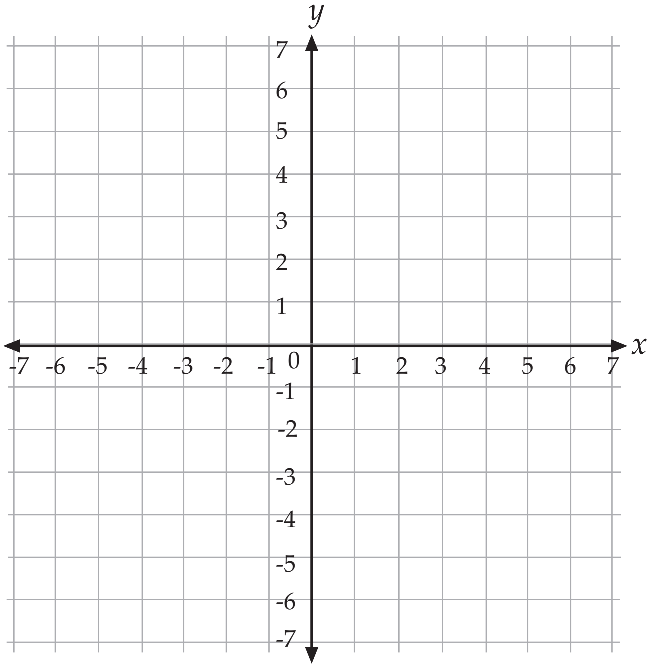
__________ 3. (-3, 0)

__________ 4. (1, 8)

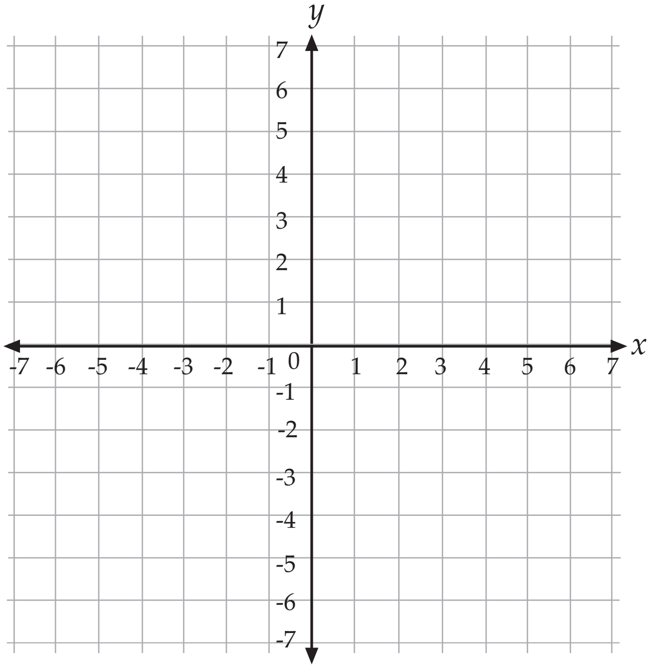
5. Find three solutions for the equation $2x + y = 6$.

____________ , ____________ , ____________

6. Graph $y = x + 4$.

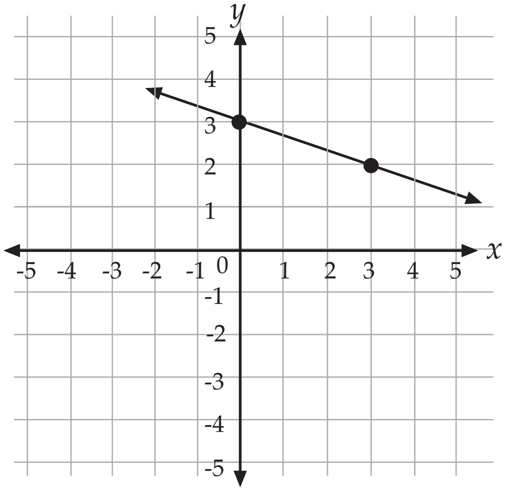


7. Graph $y = \frac{1}{3}x + 1$.



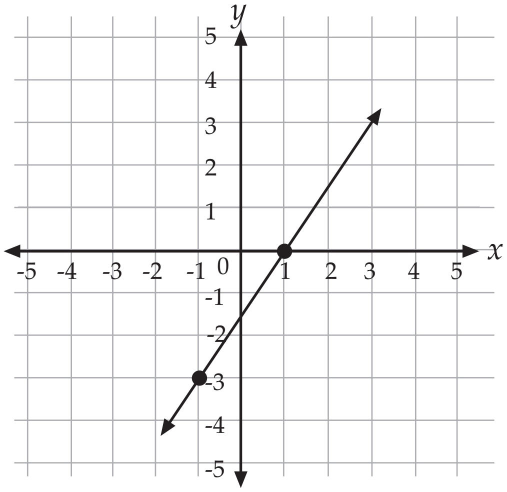
Find the **slope** *of these lines.*

8.



slope: ____________

9.



slope: ____________

Use the following **definition of slope**:

$$\text{slope} = \frac{y_2 - y_1}{x_2 - x_1}$$

10. Find the slope of the lines passing through (-6, 5) and (8, -4).

____

11. Find the slope of the lines passing though (0, -2) and (8, -4).

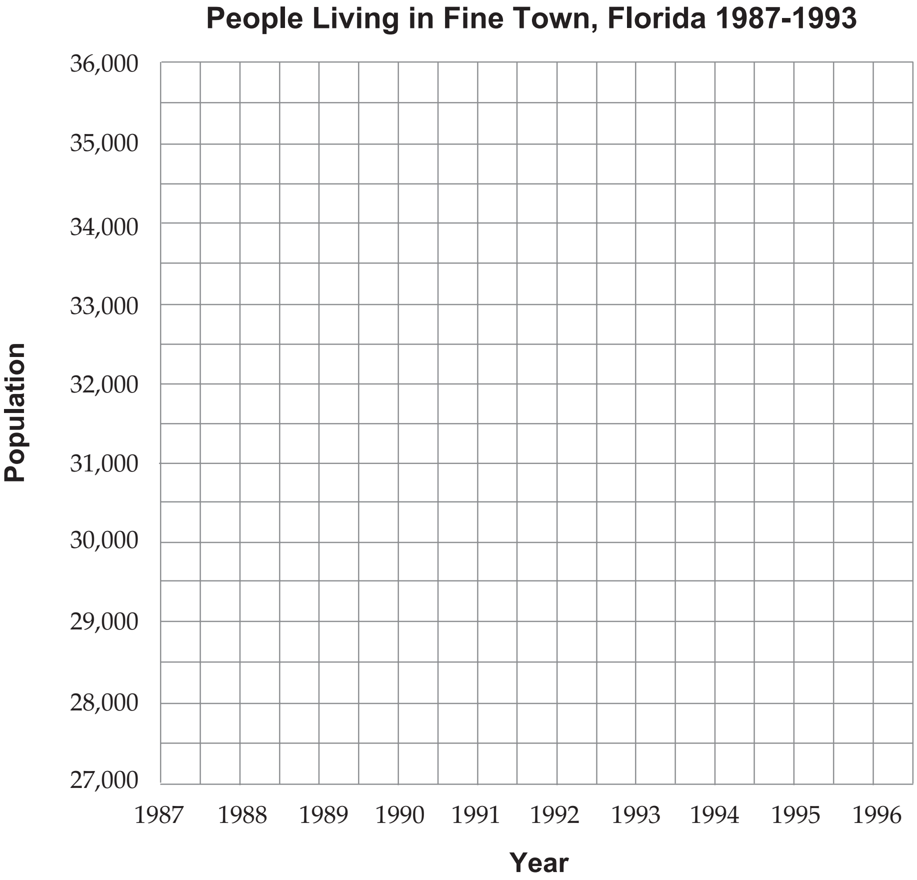
____

12. Below is a table which gives the number of people living in Fine Town, Florida from 1987 through 1993.

People Living in Fine Town, Florida 1987-1993

Year	Population
1987	27,000
1988	28,000
1989	28,920
1990	30,000
1991	30,840
1992	31,900
1993	32,760

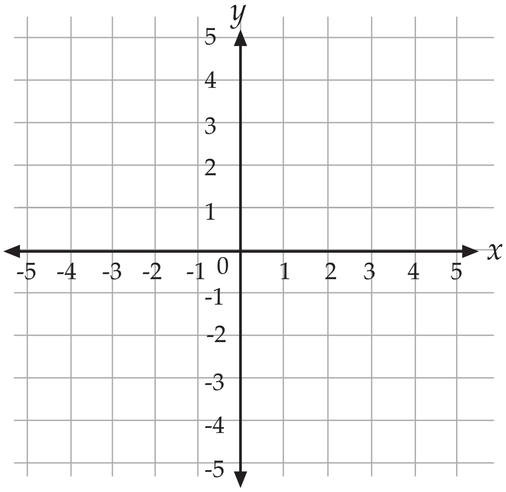
Plot the points and then draw a line of best fit. Use it to estimate the population in 1995.



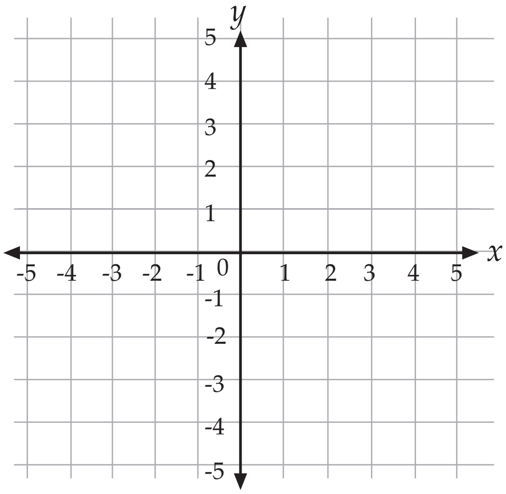
Estimate: ______________________________________

Graph the following **inequalities**.

13. $y < 2x + 1$



14. $y \geq x - 4$



Write the **next three numbers** *of each* **sequence**.

15. 0, 3, 6, 9, ____________ , ____________ , ____________

16. 1, $1\frac{1}{4}$, $1\frac{1}{2}$, $1\frac{3}{4}$, ____________ , ____________ , ____________

Answer the following.

17. Find the 8th term in the sequence.

    100, 102, 104, 106, … ____________

18. Fill in the missing numbers for row 5.

    1
    1 1
    1 2 1
    1 3 3 1
    1 __ 6 __ 1

19. Using the formula $180(n-2)$, find the number of degrees in a regular hexagon.

20. Using the formula $\frac{180(n-2)}{n}$, find the measure of each angle in a regular hexagon.
    Hint: A regular hexagon has all sides and angles equal.

21. Janet has 3 skirts (blue, tan, and black), and 4 blouses (blue, white, pink, and yellow). Draw a tree diagram to show how many different outfits she can wear.

22. Eduardo is buying a car. He can choose from 3 models, each offered in 5 different colors, and either a manual or automatic transmission. How many buying options are available?

Answer the following. Write each answer in **simplest form***.*

23. Tawanna puts 50 numbers from 1 to 50 in a glass bowl. What would be the probability of drawing an

    a. even number? ______________________________

    b. a multiple of 5? ______________________________

    Remember: A **multiple** is the number that results from multiplying a given number by the set of whole numbers. For example, *multiples* of 15 are 0, 15, 30, 45, 60, 75, etc.

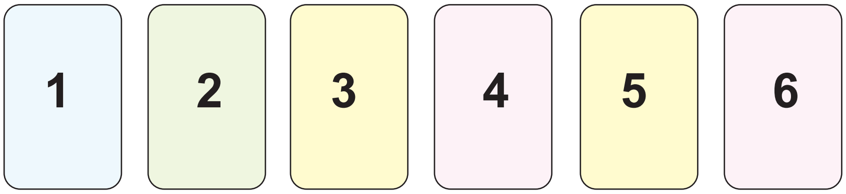
24. Dan draws 1 card from a standard deck of 52 playing cards.

    a. Find the probability of drawing a 5. ________________

    b. Find the probability of drawing a jack or a 10. __________

25. Wanda has 4 pairs of shorts (black, blue, tan, and brown) and 6 pairs of socks (black, blue, tan, brown, pink, and purple). If she reaches into her shorts drawer and then into her sock drawer without looking, what would be the probability that she will have a tan pair of shorts and tan socks?

26. Consider drawing a card from the cards shown below, and then, *without* replacement, drawing a second card. Find the probability of drawing a 5 and then a 3. The cards will be face down and will have been shuffled.



Numbers 27-29 are **gridded-response items**.
Write answers along the top of the grid and correctly mark them below. **Round to nearest whole number.**

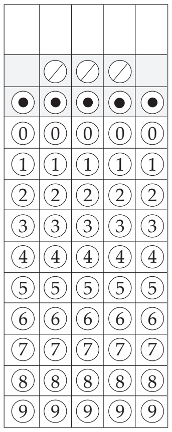
Below are Jared's pre-algebra grades for the nine weeks:

50, 68, 70, 70, 70, 70, 85, 85, and 90.

Find the following:

27. mean

Mark your answer on the grid to the right.



28. mode

Mark your answer on the grid to the right.

	⊘	⊘	⊘	
⊙	⊙	⊙	⊙	⊙
⓪	⓪	⓪	⓪	⓪
①	①	①	①	①
②	②	②	②	②
③	③	③	③	③
④	④	④	④	④
⑤	⑤	⑤	⑤	⑤
⑥	⑥	⑥	⑥	⑥
⑦	⑦	⑦	⑦	⑦
⑧	⑧	⑧	⑧	⑧
⑨	⑨	⑨	⑨	⑨

29. median

Mark your answer on the grid to the right.

	⊘	⊘	⊘	
⊙	⊙	⊙	⊙	⊙
⓪	⓪	⓪	⓪	⓪
①	①	①	①	①
②	②	②	②	②
③	③	③	③	③
④	④	④	④	④
⑤	⑤	⑤	⑤	⑤
⑥	⑥	⑥	⑥	⑥
⑦	⑦	⑦	⑦	⑦
⑧	⑧	⑧	⑧	⑧
⑨	⑨	⑨	⑨	⑨

30. range ______________

Appendices

Table of Squares and Approximate Square Roots

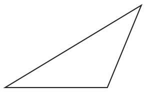
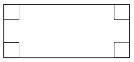
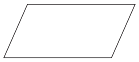
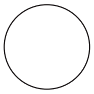
n	n^2	$\sqrt{n}$
1	1	1.000
2	4	1.414
3	9	1.732
4	16	2.000
5	25	2.236
6	36	2.449
7	49	2.646
8	64	2.828
9	81	3.000
10	100	3.162
11	121	3.317
12	144	3.464
13	169	3.606
14	196	3.742
15	225	3.873
16	256	4.000
17	289	4.123
18	324	4.243
19	361	4.359
20	400	4.472
21	441	4.583
22	484	4.690
23	529	4.796
24	576	4.899
25	625	5.00
26	676	5.099
27	729	5.196
28	784	5.292
29	841	5.385
30	900	5.477
31	961	5.568
32	1,024	5.657
33	1,089	5.745
34	1,156	5.831
35	1,225	5.916
36	1,296	6.000
37	1,369	6.083
38	1,444	6.164
39	1,521	6.245
40	1,600	6.325
41	1,681	6.403
42	1,764	6.481
43	1,849	6.557
44	1,936	6.633
45	2,025	6.708
46	2,116	6.782
47	2,209	6.856
48	2,304	6.928
49	2,401	7.000
50	2,500	7.071
51	2,601	7.141
52	2,704	7.211
53	2,809	7.280
54	2,916	7.348
55	3,025	7.416
56	3,136	7.483
57	3,249	7.550
58	3,364	7.616
59	3,481	7.681
60	3,600	7.746
61	3,721	7.810
62	3,844	7.874
63	3,969	7.937
64	4,096	8.000
65	4,225	8.062
66	4,356	8.124
67	4,489	8.185
68	4,624	8.246
69	4,761	8.307
70	4,900	8.367
71	5,041	8.426
72	5,184	8.485
73	5,329	8.544
74	5,476	8.602
75	5,625	8.660
76	5,776	8.718
77	5,929	8.775
78	6,084	8.832
79	6,241	8.888
80	6,400	8.944
81	6,561	9.000
82	6,724	9.055
83	6,889	9.110
84	7,056	9.165
85	7,225	9.220
86	7,396	9.274
87	7,569	9.327
88	7,744	9.381
89	7,921	9.434
90	8,100	9.487
91	8,281	9.539
92	8,464	9.592
93	8,649	9.644
94	8,836	9.695
95	9,025	9.747
96	9,216	9.798
97	9,409	9.849
98	9,604	9.899
99	9,801	9.950
100	10,000	10.000

Mathematical Symbols

÷ *or* /	divide	∥	is parallel to
x *or* •	times	$\approx$	is approximately equal to
$=$	is equal to	$\cong$	is congruent to
$-$	negative	$\sim$	is similar to
$+$	positive	$\sqrt{}$	nonnegative square root
$\pm$	positive or negative	%	percent
$\neq$	is not equal to	π	pi
$>$	is greater than	$\overleftrightarrow{AB}$	line *AB*
$<$	is less than	$\overline{AB}$	line segment *AB*
$\ngtr$	is not greater than	$\overrightarrow{AB}$	ray *AB*
$\nless$	is not less than	$\triangle ABC$	triangle *ABC*
$\geq$	is greater than or equal to	$\angle ABC$	angle *ABC*
$\leq$	is less than or equal to	$m\overline{AB}$	measure of line segment *AB*
°	degrees	$m\angle ABC$	measure of angle *ABC*
$\perp$	is perpendicular to		

FCAT Mathematics Reference Sheet

Area

	**triangle**	$A = \frac{1}{2}bh$
	**rectangle**	$A = lw$
	**trapezoid**	$A = \frac{1}{2}h(b_1 + b_2)$
	**parallelogram**	$A = bh$
	**circle**	$A = \pi r^2$

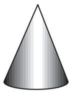
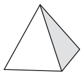
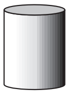
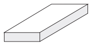
Key

b = base
h = height
l = length
w = width
ℓ = slant height
$S.A.$ = surface area
d = diameter
r = radius
A = area
C = circumference
V = volume

Use 3.14 or $\frac{22}{7}$ for π.

circumference

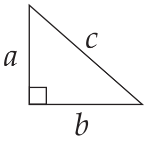
$C = \pi d$ or $C = 2\pi r$

		Volume	Total Surface Area
	**right circular cone**	$V = \frac{1}{3}\pi r^2 h$	$S.A. = \frac{1}{2}(2\pi r)\ell + \pi r^2$ or $S.A. = \pi r\ell + \pi r^2$
	**square pyramid**	$V = \frac{1}{3}lwh$	$S.A. = 4(\frac{1}{2}l\ell) + l^2$ or $S.A. = 2l\ell + l^2$
	**sphere**	$V = \frac{4}{3}\pi r^3$	$S.A. = 4\pi r^2$
	**right circular cylinder**	$V = \pi r^2 h$	$S.A. = 2\pi rh + 2\pi r^2$
	**rectangular solid**	$V = lwh$	$S.A. = 2(lw) + 2(hw) + 2(lh)$

In the following formulas, n represents the number or sides.

- In a polygon, the sum of the measures of the interior angles is equal to $180(n - 2)$.
- In a regular polygon, the measure of an interior angle is equal to $\frac{180(n-2)}{n}$.

FCAT Mathematics Reference Sheet

Pythagorean theorem:  $a^2 + b^2 = c^2$	Distance between two points $P_1\ (x_1, y_1)$ and $P_2\ (x_2, y_2)$: $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
Slope-intercept form of an equation of a line: $y = mx + b$ where m = slope and b = the y-intercept.	Midpoint between two points $P_1\ (x_1, y_1)$ and $P_2\ (x_2, y_2)$: $\left(\frac{x_2 + x_1}{2}, \frac{y_2 + y_1}{2}\right)$
Distance, rate, time formula: $d = rt$ where d = distance, r = rate, t = time.	Simple interest formula: $\mathrm{I} = prt$ where p = principal, r = rate, t = time.

Conversions

1 yard = 3 feet = 36 inches
1 mile = 1,760 yards = 5,280 feet
1 acre = 43,560 square feet
1 hour = 60 minutes
1 minute = 60 seconds

1 liter = 1000 milliliters = 1000 cubic centimeters
1 meter = 100 centimeters = 1000 millimeters
1 kilometer = 1000 meters
1 gram = 1000 milligrams
1 kilogram = 1000 gram

1 cup = 8 fluid ounces
1 pint = 2 cups
1 quart = 2 pints
1 gallon = 4 quarts

1 pound = 16 ounces
1 ton = 2,000 pounds

Metric numbers with four digits are presented without a comma (e.g., 9960 kilometers). For metric numbers greater than four digits, a space is used instead of a comma (e.g., 12 500 liters).

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Production Software

Adobe PageMaker 6.5. Mountain View, CA: Adobe Systems.

Adobe Photoshop 5.0. Mountain View, CA: Adobe Systems.

Macromedia Freehand 8.0. San Francisco: Macromedia.

Microsoft Office 98. Redmond, WA: Microsoft.